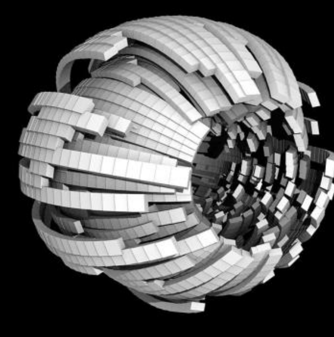
What are our words worth?

"I'll give you a thousand words for it."

Algebra as Area: Expressions with Polynomials & Equations

David Mattoon

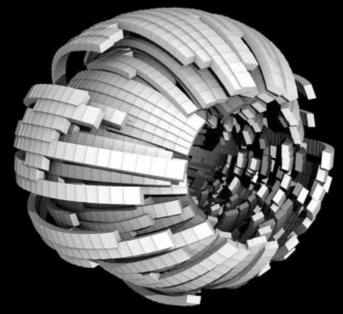
Meaning for Memory



2018 CMC-South Annual Conference

Deficiency
Time
Rigor

David Mattoon
Meaning for Memory



Deficiency: Students or Environment?

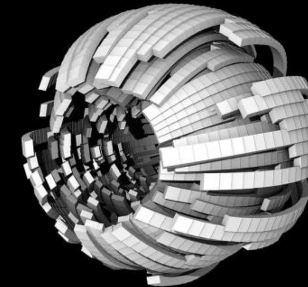
In educational settings, this construction of dis/ability manifests in the double education system that splits general education and special education. Scholars have traced the ways in which special education "**serves as a vehicle for <u>preserving general education in the midst of ever increasing diversity</u>**" (Reid & Valle, 2004, p. 468, paraphrasing Dudley-Marling, 2001; also see Skrtic, 1991, 2005). **Rather than using research-validated frameworks** like Universal Design for Learning (UDL) and Complex Instruction (CI) **to deliver rigorous, high-cognitive demand instruction to all mathematics students**, the system of special education **shunts certain students (especially students of color) into an inferior, segregated mathematics education, thus providing a <u>band-aid to a broken general education system</u> and <u>preventing larger, more systematic changes</u>.**

Returning to the assumptions inherent in the concept of intervention, a disability studies perspective problematizes **the taken-for-granted assumption that what is "wrong" with the situation requiring intervention is a pathology or deficit within students. Instead, the problem is located in the <u>inaccessibility of the environment</u>; in other words, what needs to be changed is not the student, but rather the environment to allow access for students who differ from one another.** As Reid and Valle (2004) assert, "the responsibility for 'fitting in' has more to do with changing public attitudes and the development of welcoming classroom communities and with compensatory and differentiated instructional approaches than with individual learners (Shapiro, 1999). In other words, **our focus is on <u>redesigning the context</u>, not on 'curing' or 'remediating' individuals' impairments"** (p. 468). A related line of research of working group members involves conceptualizing interventions into participation rather than content. That is, **what interventions might contribute to <u>more equitable participation and deeper engagement</u> across students in mathematics classrooms?** For example, one of the working group members has conducted empirical research focused on equitable participation in a Cognitively Guided Instruction algebra routine. Moreover, a political/relational model suggests that inaccessibility is embedded in the context of power relations. **<u>Finding ways to "intervene" to make the environment accessible</u>, then, also requires analyzing the power relations involved** in maintaining inaccessibility.

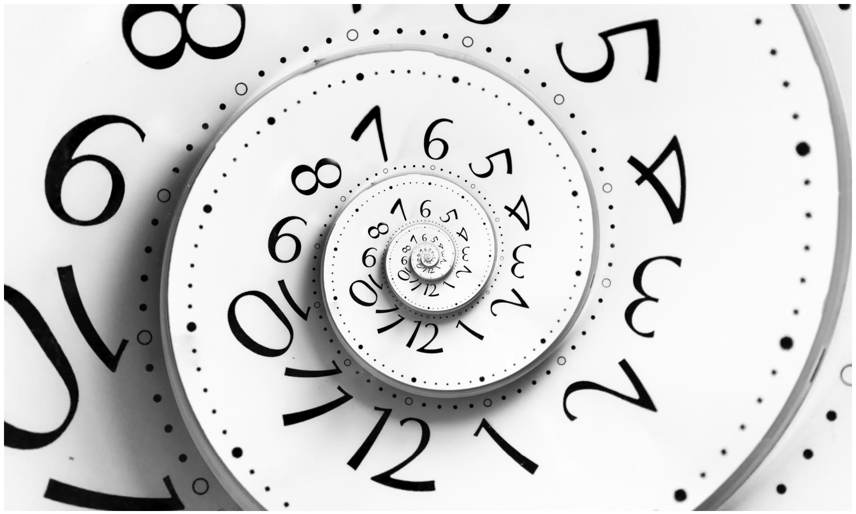
http://jamessheldon.com/reframing-interventions-in-mathematics-education-emerging-critical-perspectives/

Change the Environment

Time:
I don't have it.



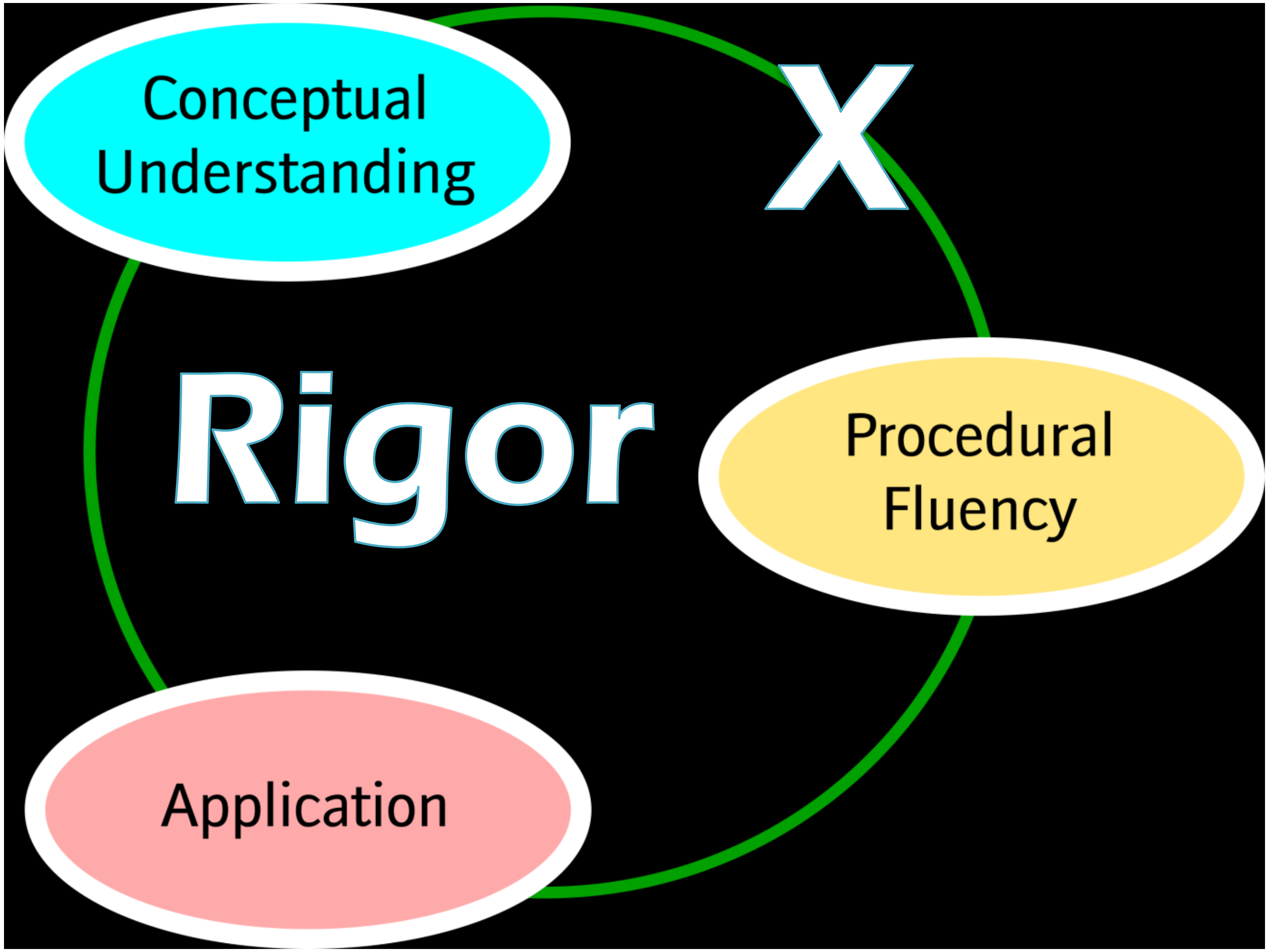
I might make time to…

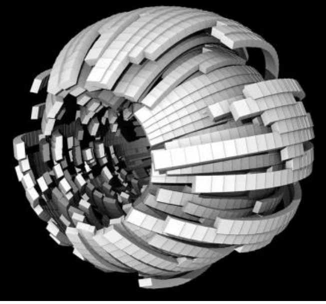


Pictorial Math will save you time in the long run as it will aid students in understanding what to do, why they do it, and how to transfer the knowledge.

- Review in the beginning of the year
- Introduce new material by re-teaching material from former grade levels
  - I am not talking about making connections to former material here.
- Reteach for full periods when they don't understand
- Reteach for full periods when they cannot remember
  - I am not talking about interleaving
- Teach multiple procedures for slightly different problems
  - I am not saying procedures are bad.

Where does this presentation live?





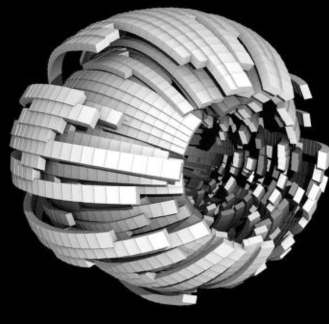
Agenda

- ✓ Deficiency, Time & Rigor
- ❑ What are Cubic , Quadratic & Linear Terms?
- ❑ Concreteness Fading
- ❑ Combining Polynomials & the Distributive Property
- ❑ Linear Expressions & Substitution
- ❑ Scaffolding? & Linear Equations

Substitution:
Volume with Surface Area Meaning of a Cubic Term

David Mattoon

Meaning for Memory



Lesson Introduction: Draw Two Congruent Cubes

On the gridded side of your whiteboard, draw three congruent cubes. (Model how to draw a cube in two dimensions)

What was the side length of your cube?

You had a side length of four.

A 4x4x4 made an volume of 64 units. Another way to write that would be 4^3. Two of them would be $2 \cdot 4^3$

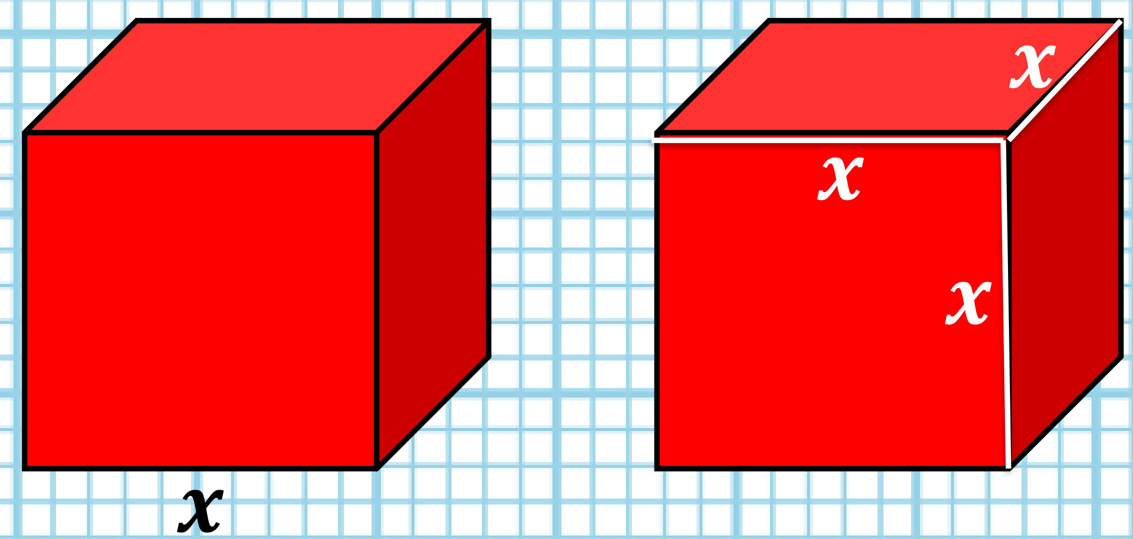
You had a side length of 10.

A 10x10x10 made an volume of 1000 units. Another way to write that would be 10^3. Two of them would be $2 \cdot 10^3$

I can draw them all !

I will draw all
of your cubes
in less than 30
seconds…

All the sides lengths in a single picture, MIRACULOUS algebra!

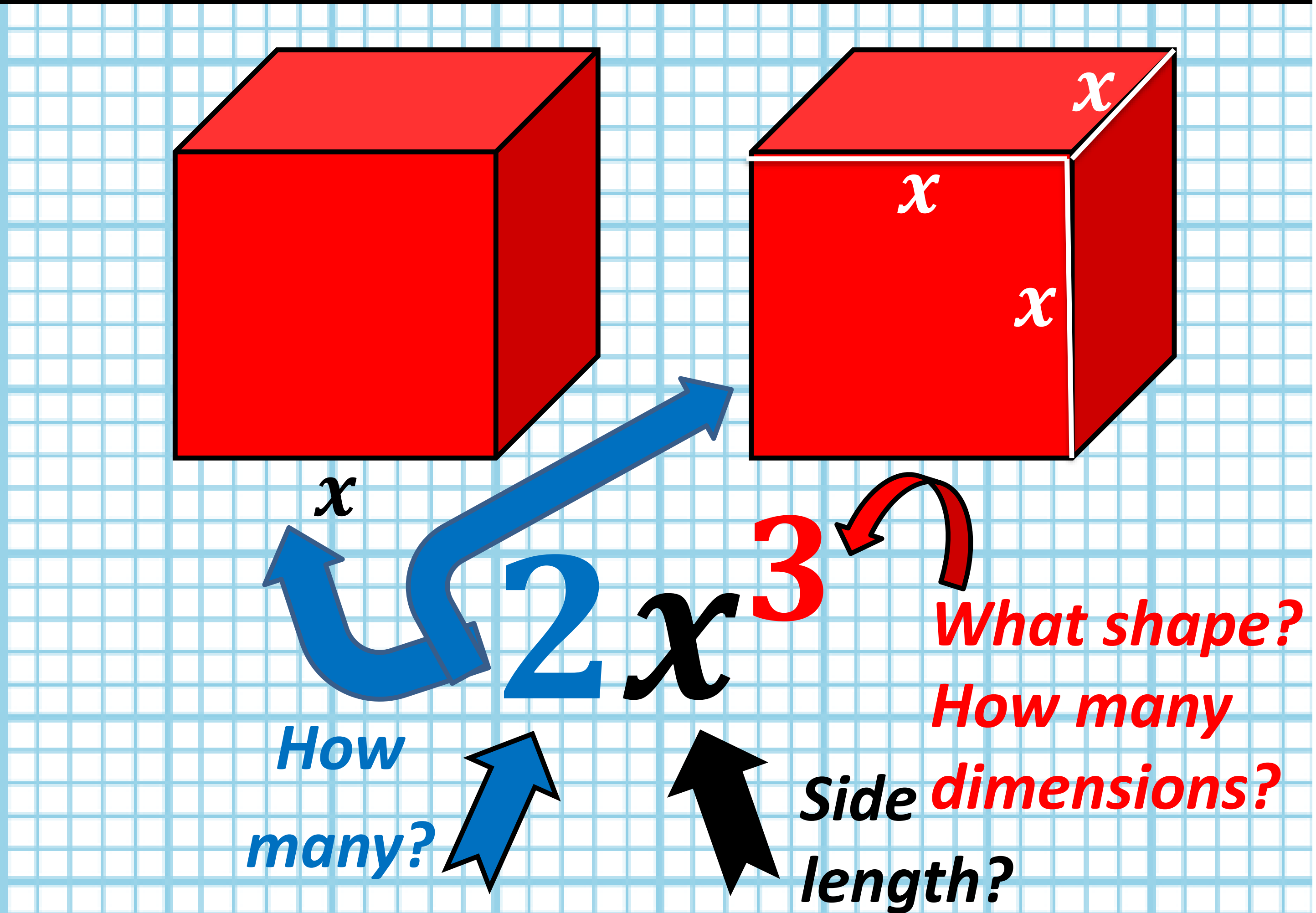


What is the length of the side of my cubes as labeled?

Any length.

Notice a cube has three dimensions length, width and height (or depth)

Anatomy of an Algebraic Term

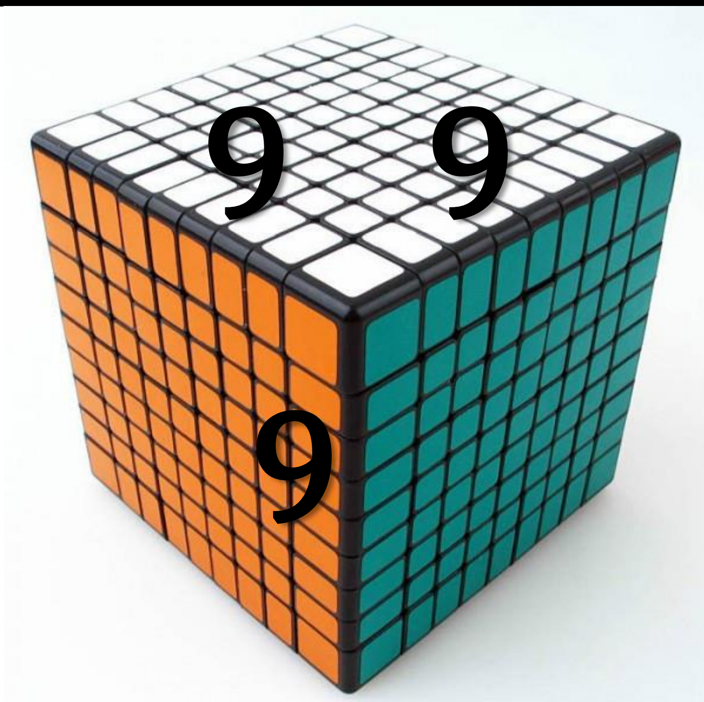


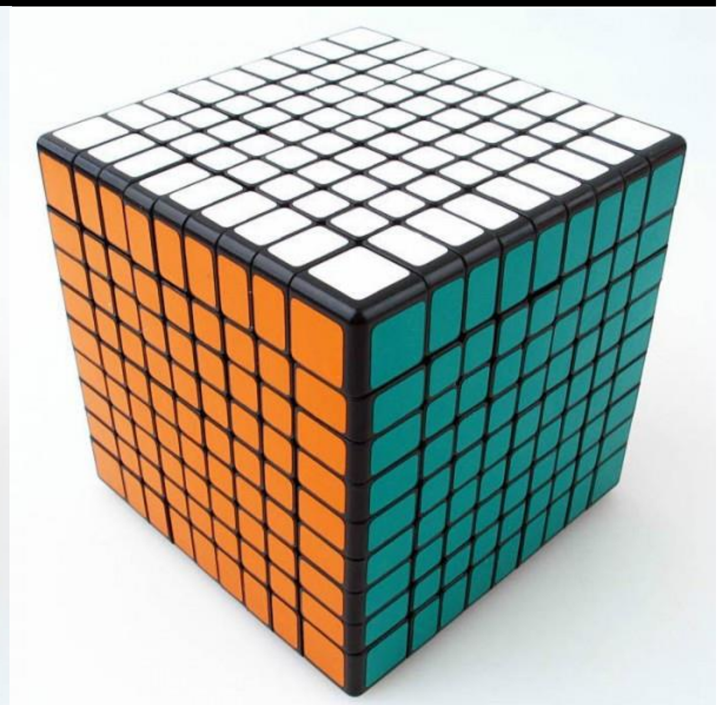
What is the length of the side of my cubes as labeled?

A $x^1 \cdot x^1 \cdot x^1$ made an volume of x^3.

Two of them would be written as $2x^3$.

Count the unit cubes.
How many cube units?

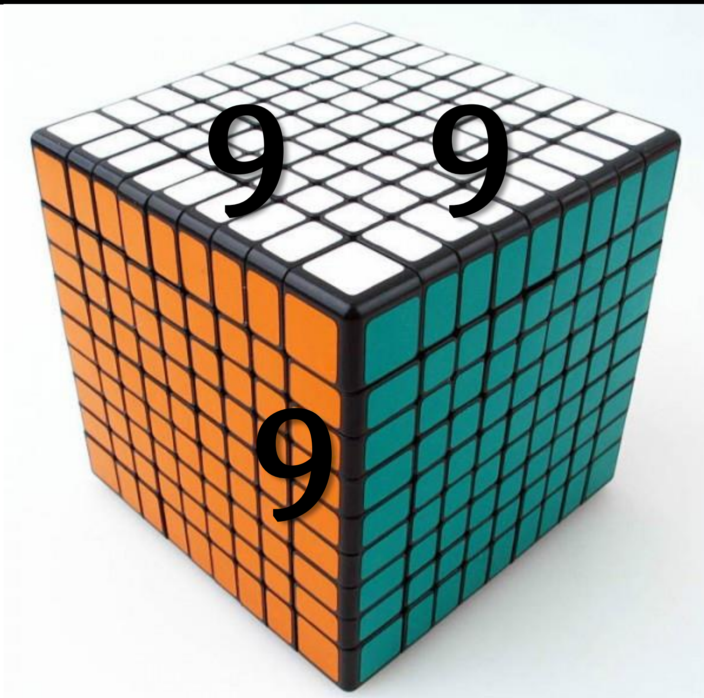


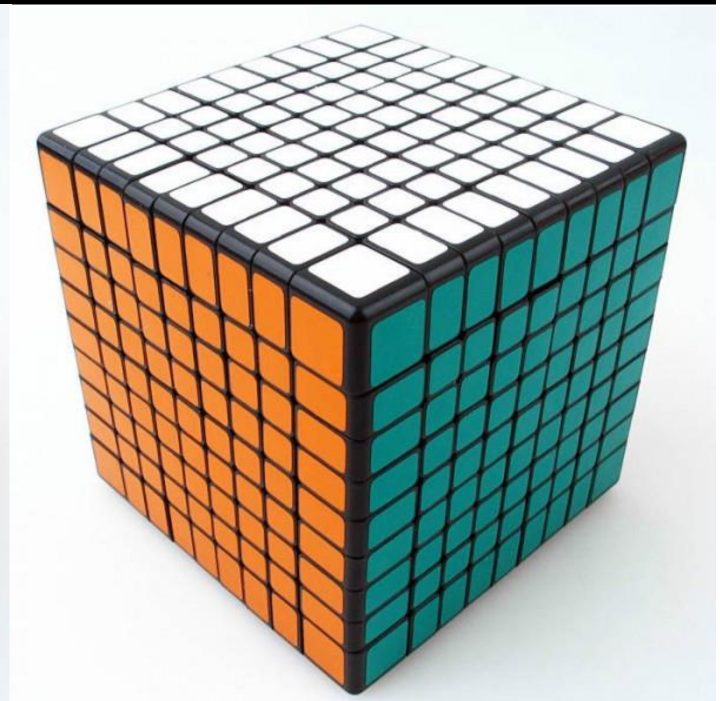


If I make x=9, then how many cube units do I have?

How did you count it? Tell a partner.

How did you count it? What does that have to do with Order of Operations?





How did you count it? One cube at a time?

$$9^3 + 9^3 = (9 \cdot 9 \cdot 9) + (9 \cdot 9 \cdot 9) = 729 + 729$$
$$= 1458$$

How did you count it?

One box at a time?

1+1+1...=1458

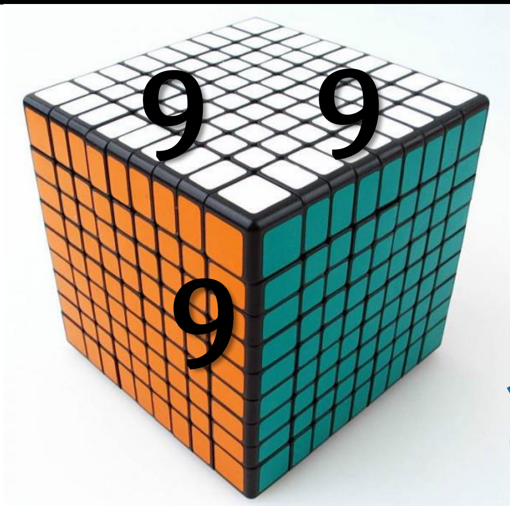
Count one square layer then multiply by nine and then two?

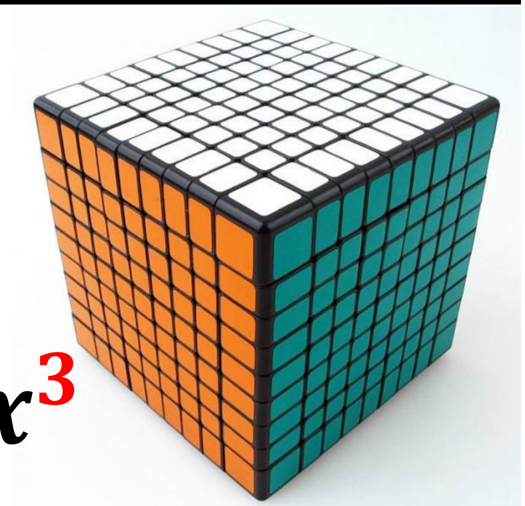
81x9x2=1458

Multiply the side lengths and then multiply by two?

(9x9x9)x2= 1458

Do you understand what you are doing in context?





$$1x^3 + 1x^3$$

$$2x^3$$

$$2(9)^3$$

$$2(729)$$

$$1458$$

If I make x=9, then how many cube units do I have?

How many in each cube, what is the volume of a single cube?

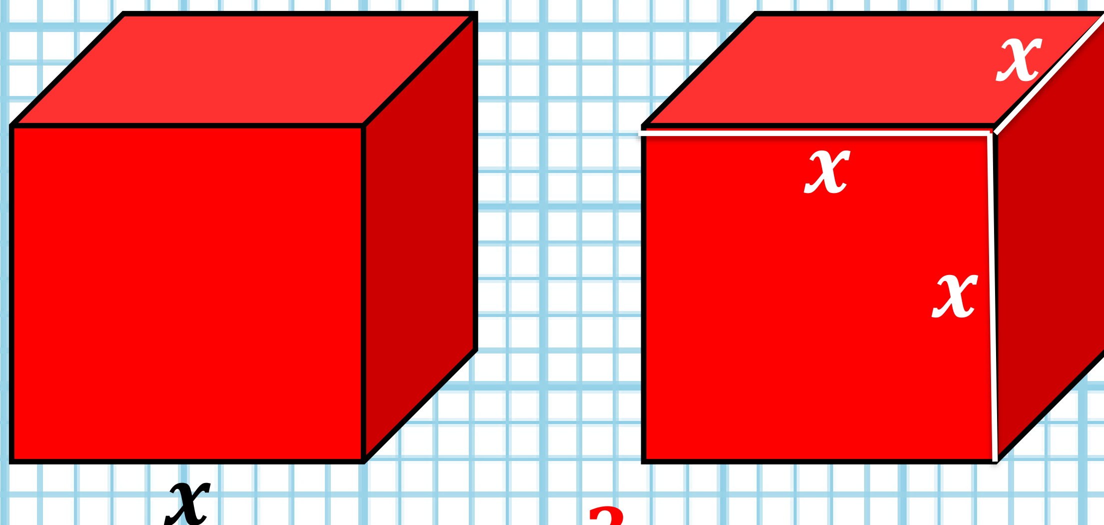
$(9)^3 = 729$

How many altogether?

$2(729) = 1458$

Notice order of operations makes sense in this volume context.

What is the volume of your two cubes?



$$2x^{3}$$

$$2(\quad)^{3}$$

$$2(\quad)$$

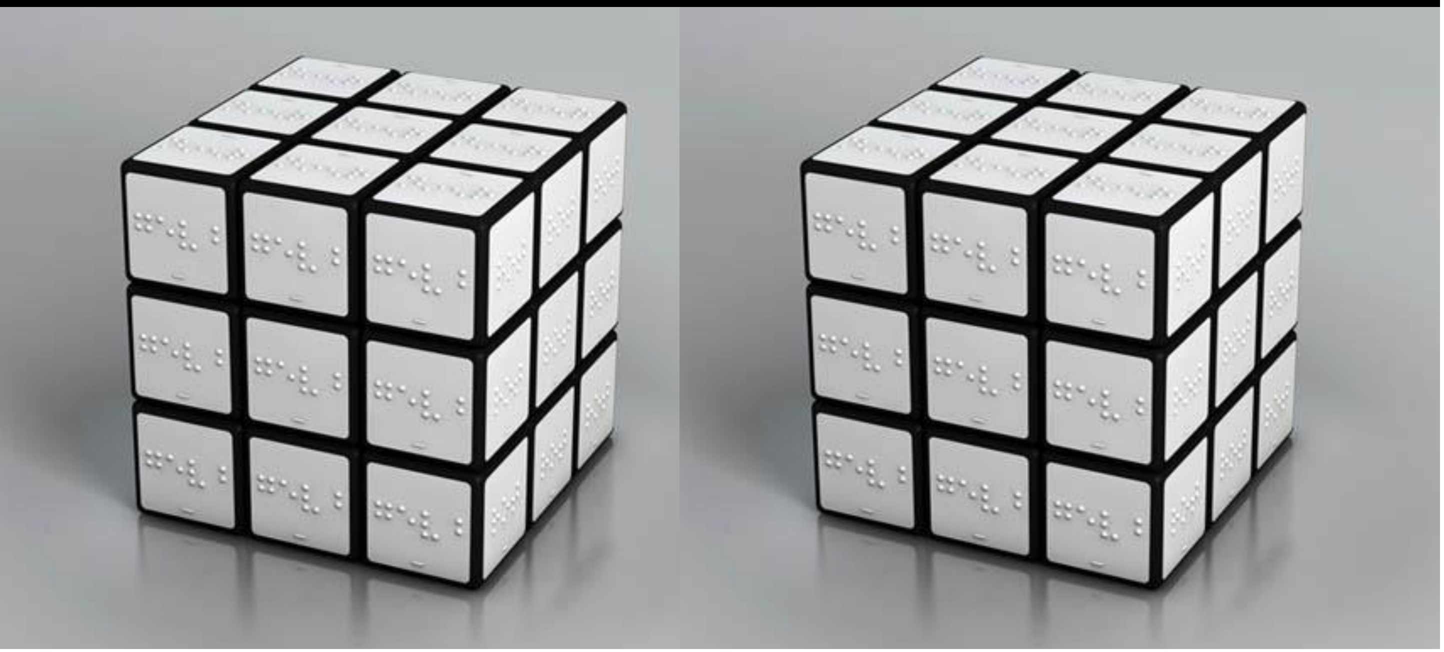
Find the volume of <u>your</u> two cubes by substituting into the algebraic expression.

Write it on the other, blank side of your whiteboard.

Show your work using proper notation like that on the left.

Have someone nearby check your work.

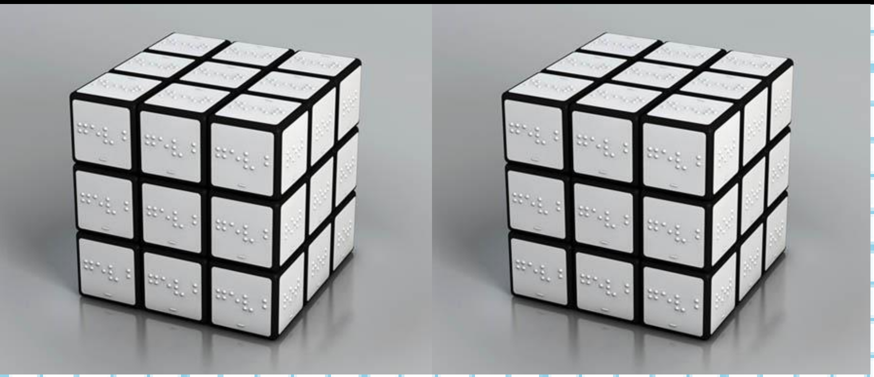
Extension: Find the Surface Area



These are two Braile rubix cubes. I need to put colored stickers on them so people who don't know Braile can do them.

How many stickers do I need?

Extension: Find the Surface Area



$$2(3^2+3^2+3^2+3^2+3^2+3^2)$$

$$2[6(3^2)]$$

$$2[6(9)]$$

$$2[54]$$

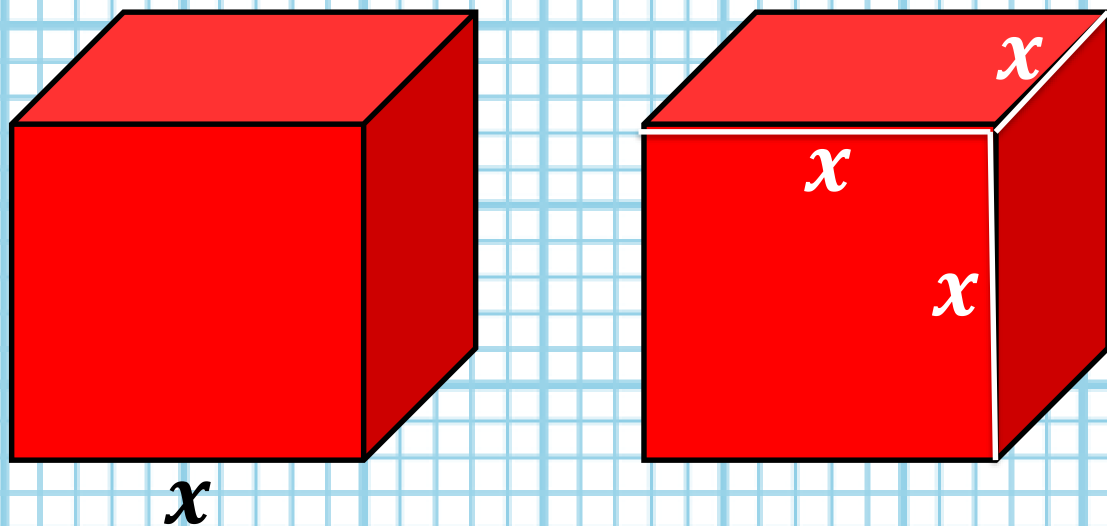
$$108$$

Is this a progression of simplifying an expression or different entry points?

Notice order of operations once again makes sense, you find the surface area of one then multiply.

Did anyone think of it like this: $12(9)=108$?

Find the Surface Area of Every Person's Two Cubes



$$2(x^2 + x^2 + x^2 + x^2 + x^2 + x^2)$$

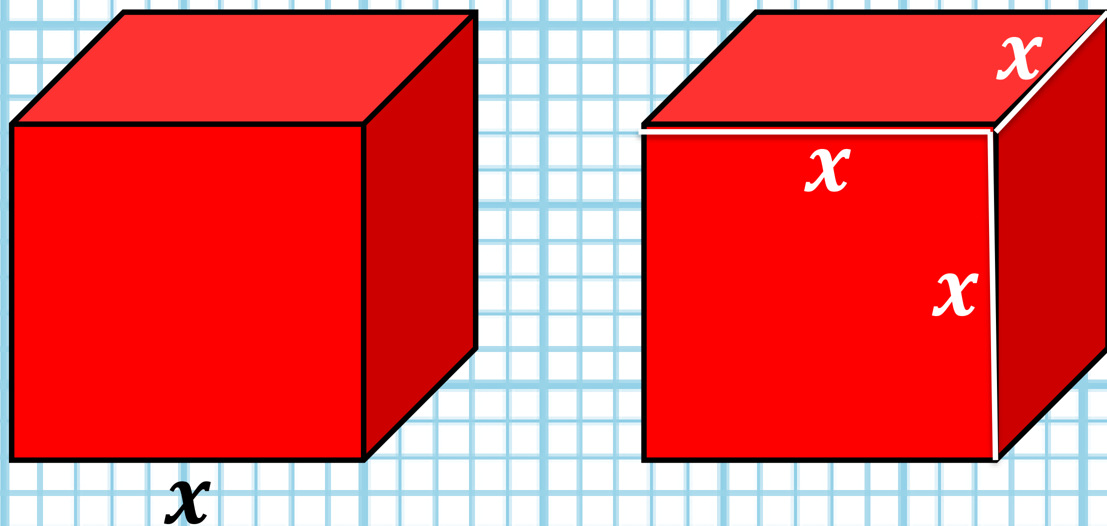
$$2(6x^2)$$

$$12x^2$$

What is the length of the side of my cubes as labeled?

Any length.

Find the Surface Area of Your Two Cubes



$$2(x^2 + x^2 + x^2 + x^2 + x^2 + x^2)$$

$$2(6x^2)$$

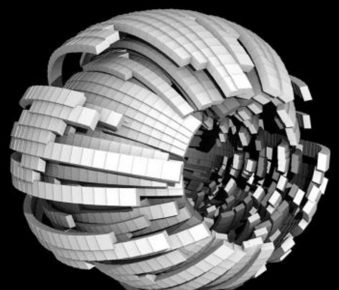
$$12x^2$$

Find the surface area of your three squares by substituting in your side length into one of the algebraic expressions. Show your work on your whiteboard.

Tell your partner which expression you picked and why?

Substitution:
Area with Perimeter
Meaning of a Quadratic Term

David Mattoon
Meaning for Memory



Lesson Introduction: Make Three Congruent Squares

On the gridded side of your whiteboard, draw three congruent squares.

What was the side length of your square?

You had a side length of four.

A 4x4 made an area of 16 units. Another way to write that would be 4^2. Three of them would be $3 \cdot 4^2$

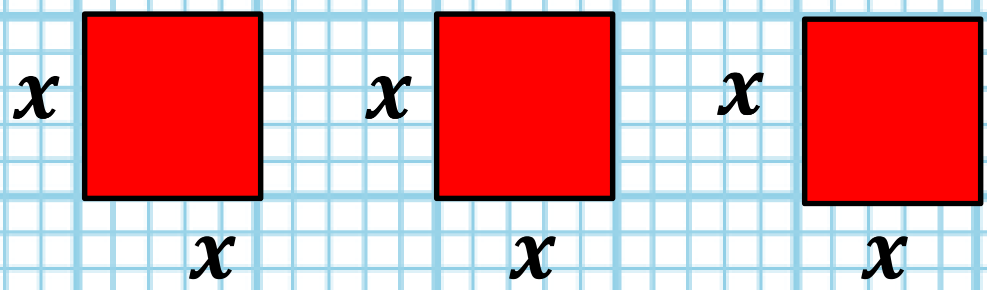
You had a side length of 10.

A 10x10 made an area of 100 units. Another way to write that would be 10^2. Three of them would be $3 \cdot 10^2$

I can draw them all !

I will draw all of your squares in less than 20 seconds...

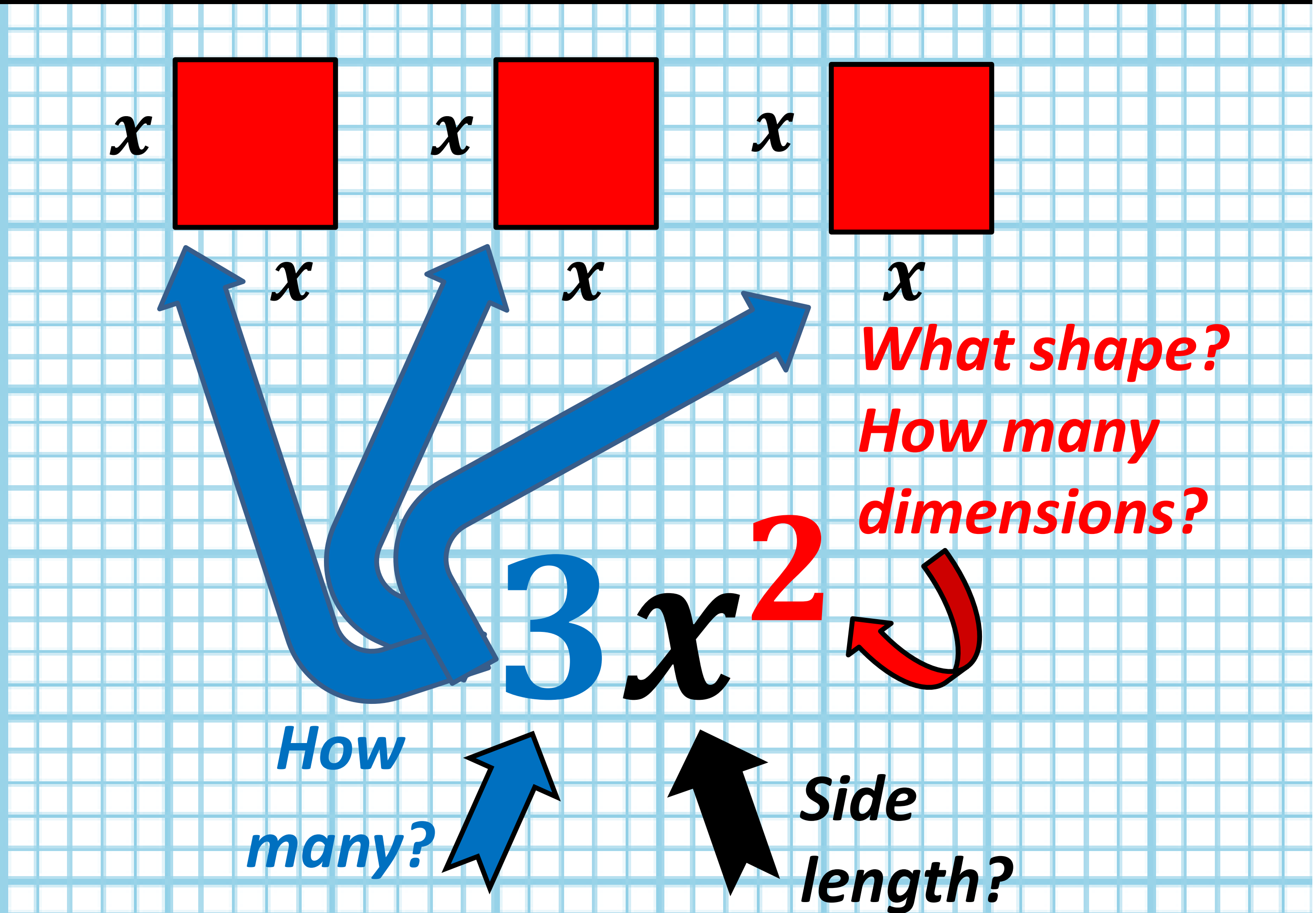
All the sides lengths in a single picture, MIRACULOUS algebra!



What is the length of the side of my *squares* as labeled?

Any length.

Anatomy of an Algebraic Term

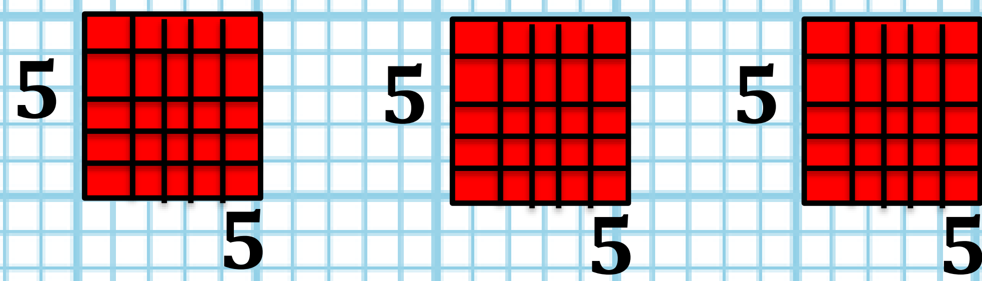


What is the length of the side of my squares as labeled?

A $x^1 \cdot x^1$ made an area of x^2.

Three of them would be written as $3x^2$.

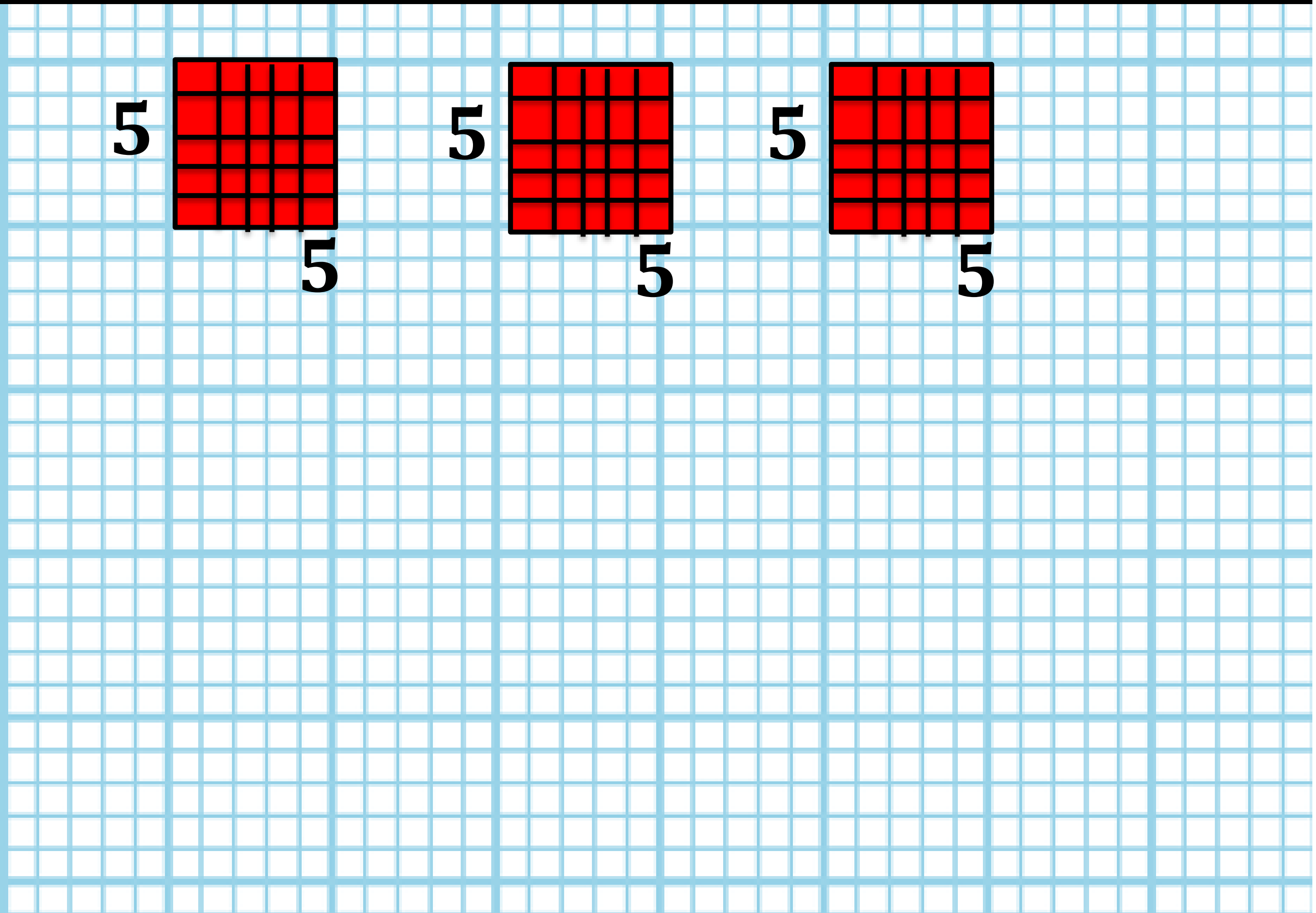
Count the unit squares.
How many square units?



If I make x=5, then how many square units do I have?

How did you count it? Tell a partner.

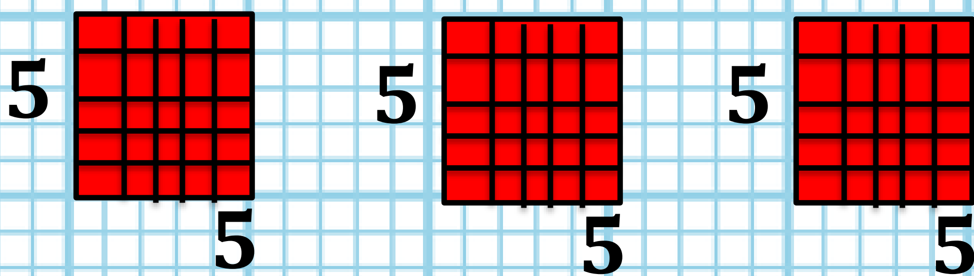
How did you count it? What does that have to do with Order of Operations?



How did you count it?

- One box at a time? $1+1+1\ldots=75$
- Count one box then multiply by three? $25x3=75$
- Multiply the side lengths and then multiply by three? $(5x5)x3=75$

Do you understand what you are doing in context?



$$1x^2 + 1x^2 + 1x^2$$

$$3x^2$$

$$3(5)^2$$

$$3(25) = 75$$

If I make x=5, then how many square units do I have?

How many in each square, what is the area of a single square?

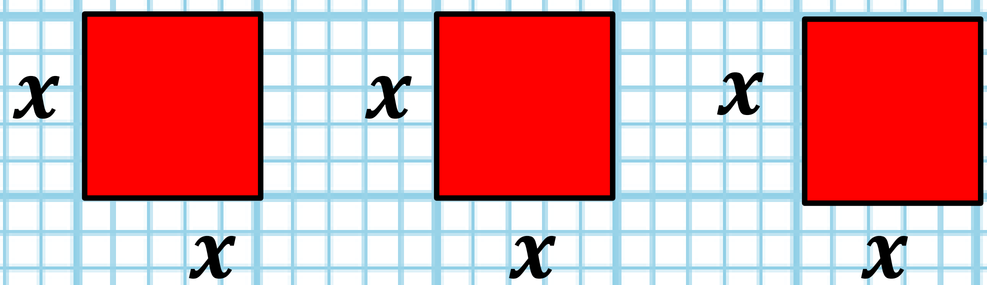
$(5)^2 = 25$

How many altogether?

$3(25) = 75$

Notice order of operations makes sense in this area context.

What is the area of your three squares?



$$3x^2$$

$$3(\quad)^2$$

$$3(\quad)$$

Find the area of <u>your</u> three squares by substituting into the algebraic expression.

Write it on the other, blank side of your whiteboard.

Show your work using proper notation like that on the left.

Have someone nearby check your work.

Substitution: Length of a Side Meaning of a Linear Term

David Mattoon

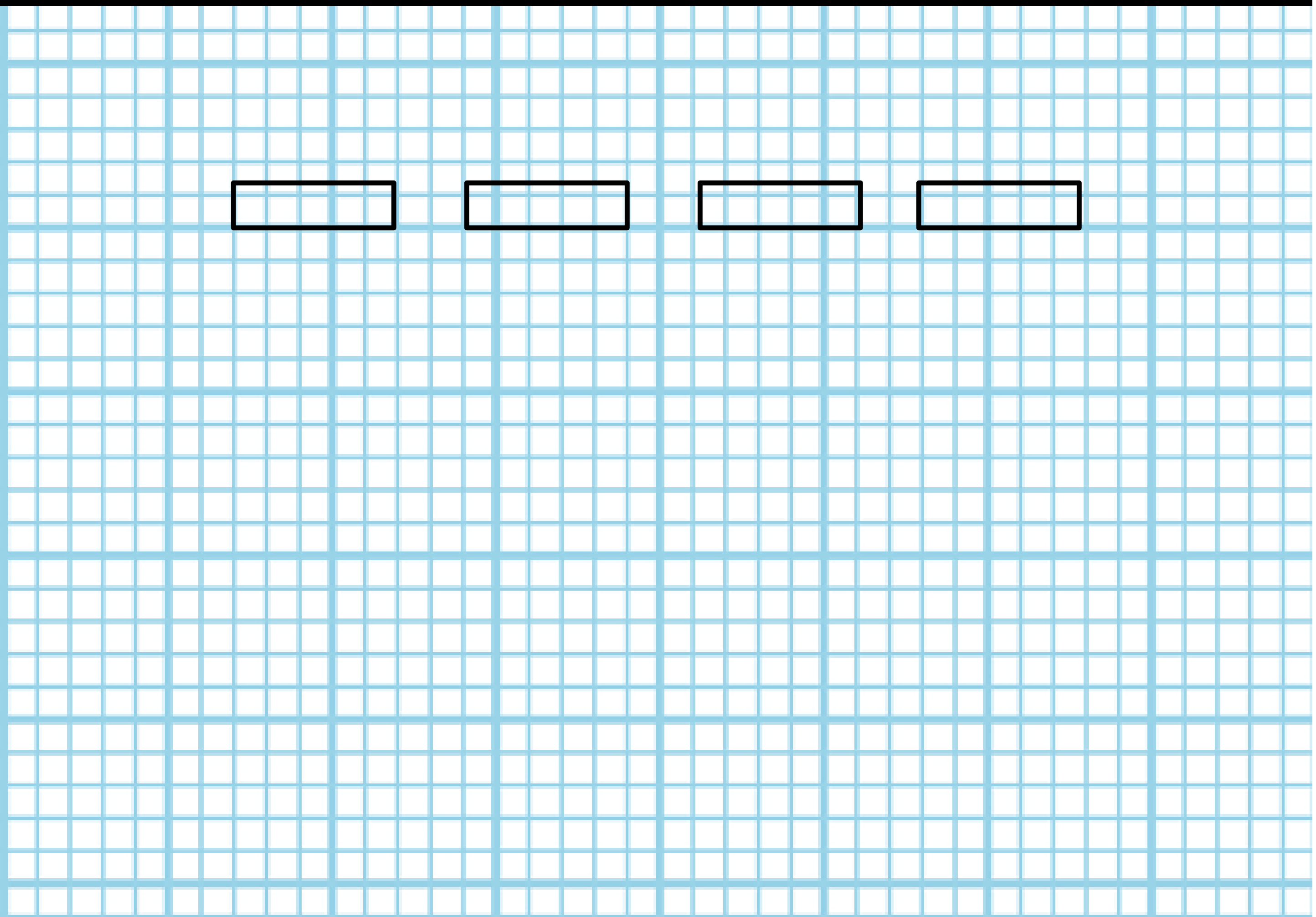
Meaning for Memory

Lesson Introduction: Make Four Congruent Lines

On the gridded side of your whiteboard, draw four congruent lines.

Leave some space around them

Lesson Introduction: Make Four Congruent Lines



Now go back and give each of your lines a height of one so we can see them better. In other words, make them into rectangles with a height of one.

What was the side length of your square?

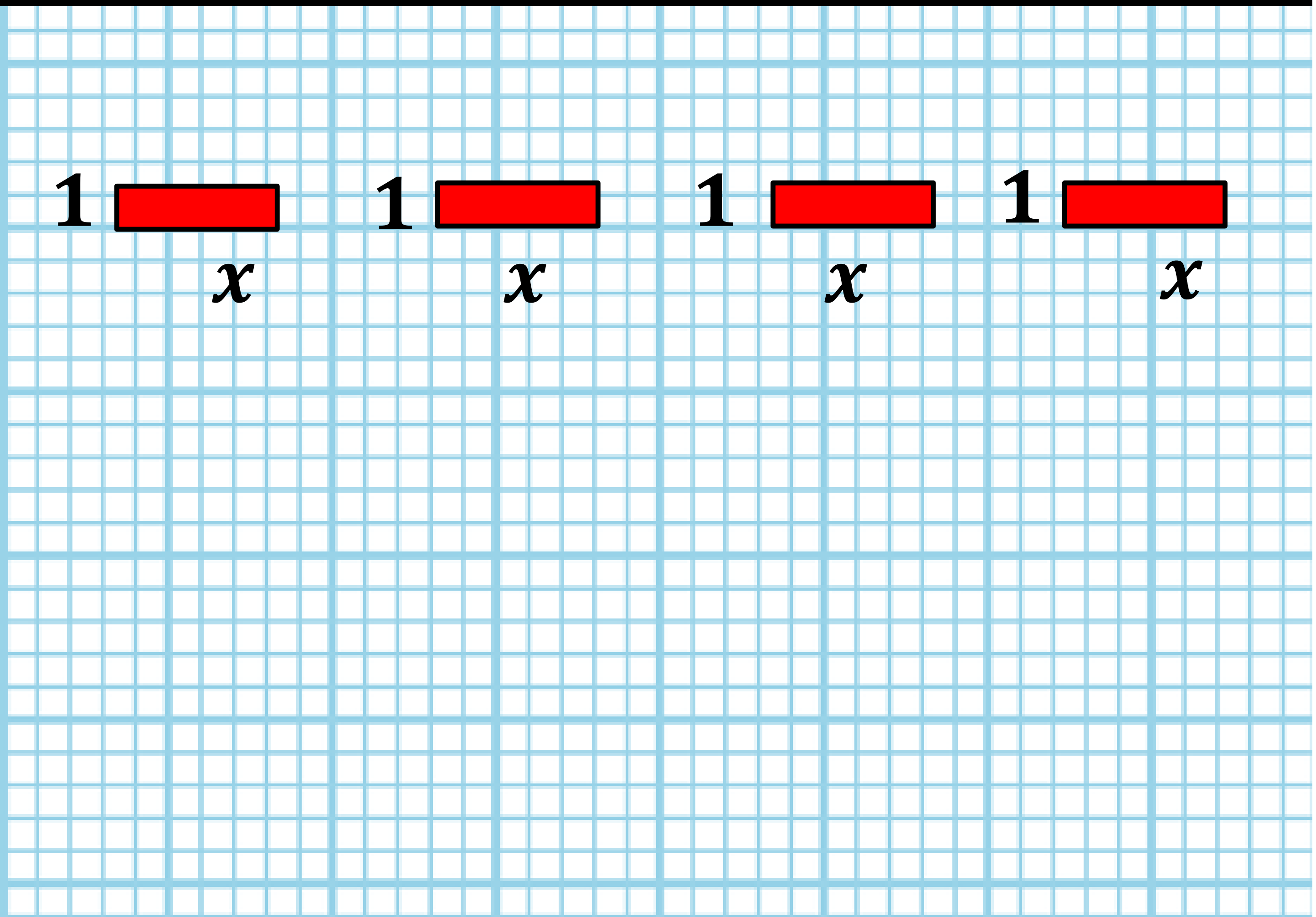
If you had a length of five, then another way to write that would be 5^1. Four of them would be $4 \cdot 5^1$

If you had a side length of 10, then another way to write that would be 10^1. Four of them would be $4 \cdot 10^1$

I can draw them all !

I will draw all of your lines (rectangles) in less than 20 seconds…

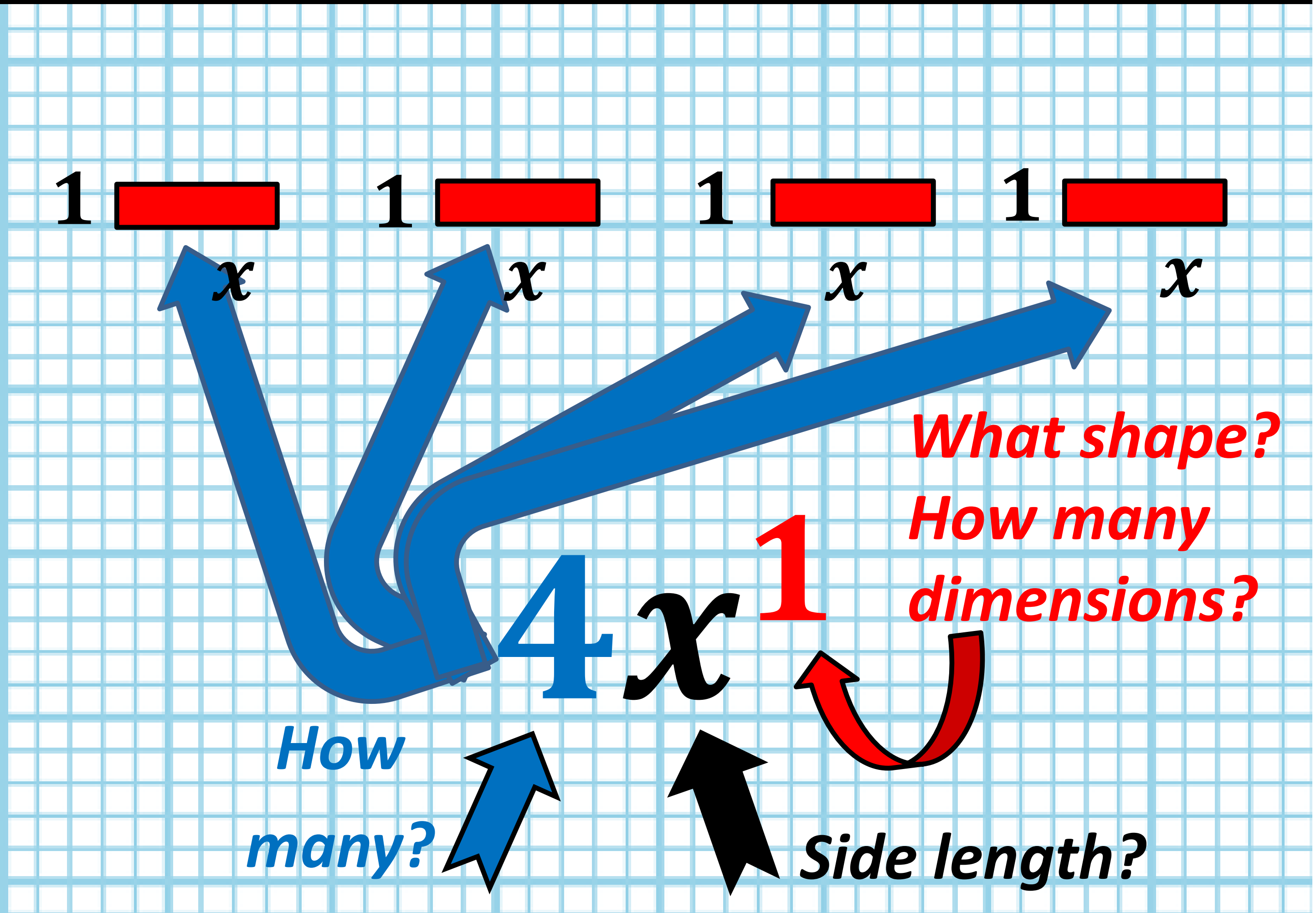
All the sides lengths in a single picture, MIRACULOUS algebra!



What is the length of the side of my lines as labeled?

Any length.

Anatomy of an Algebraic Term

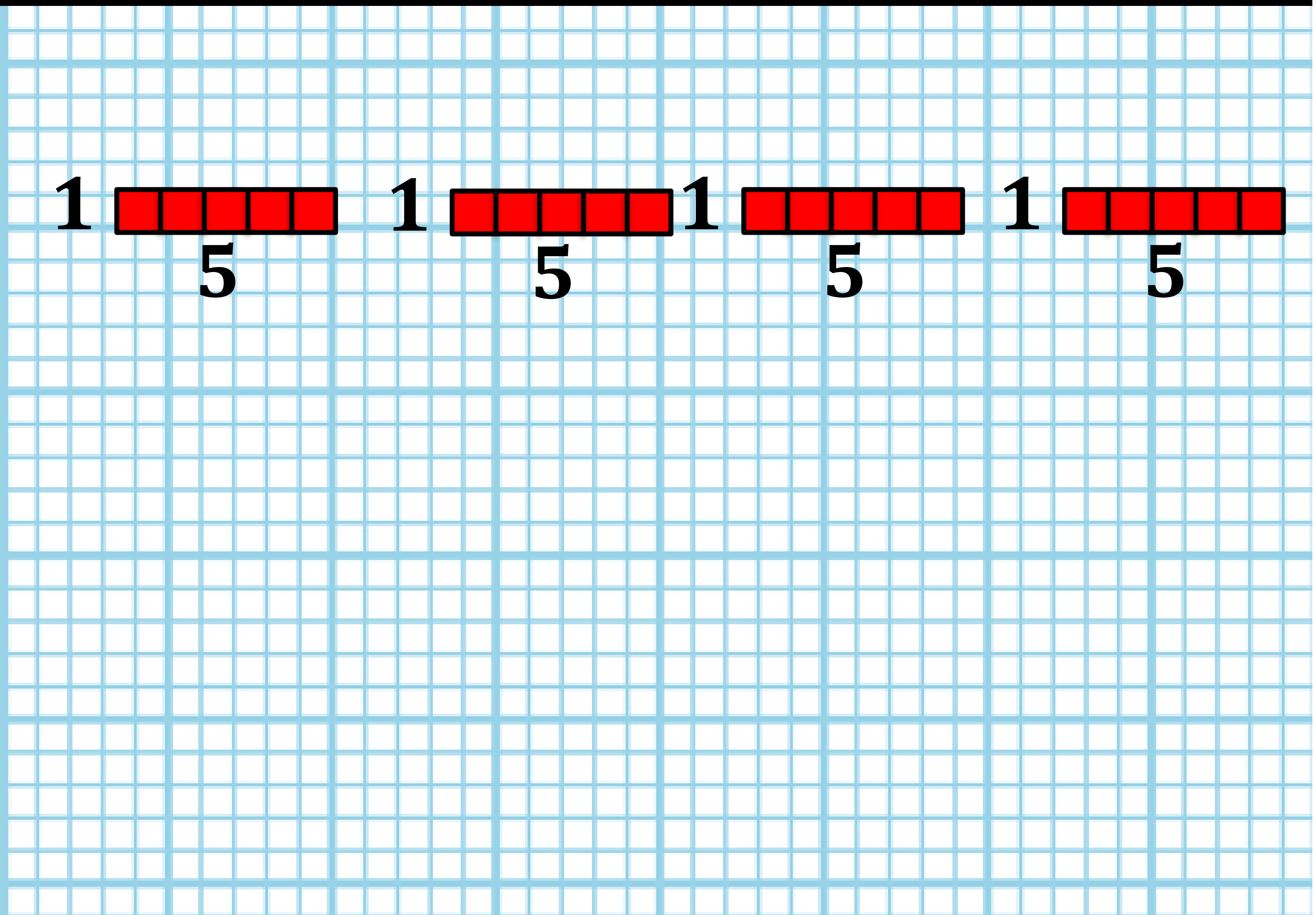


What is the length of the side of my lines as labeled?

A $1 \cdot x^1$ made an area of $1x^1$.

Four of them would be written as $4x^1$.

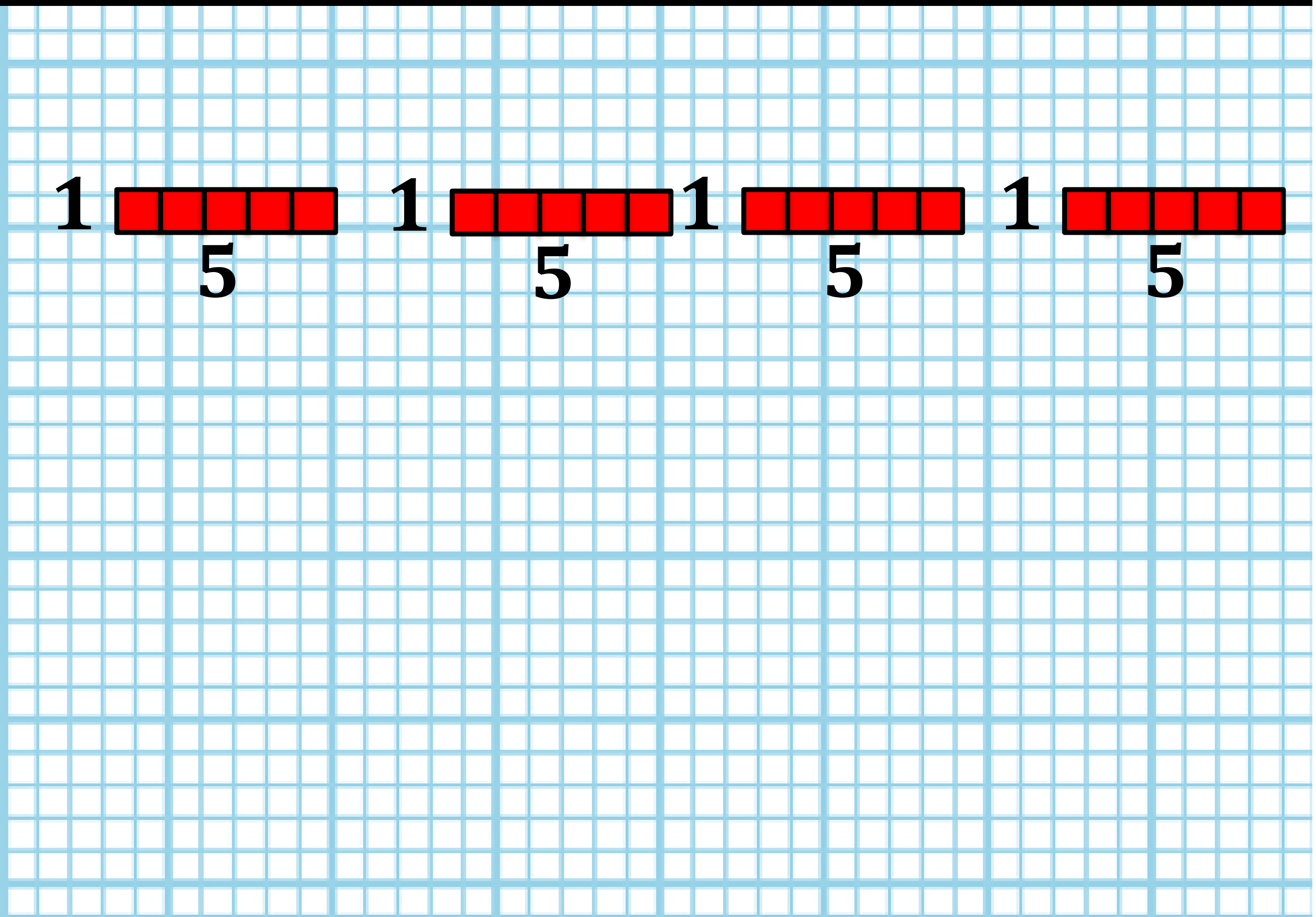
Count the unit squares. How many square units?



If I make x=5, then how many square units do I have?

How did you count it? Tell a partner.

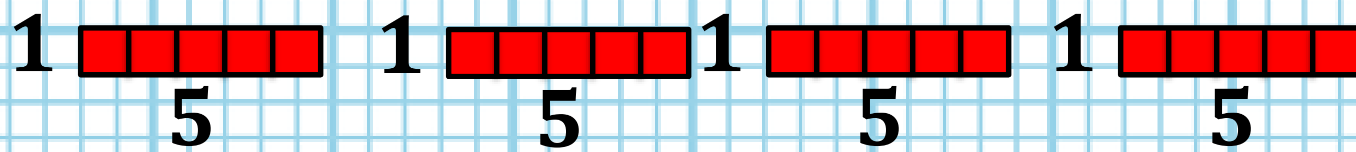
How did you count it?
What does that have to do with Order of Operations?



How did you count it?

- One box at a time?
  1+1+1…=30
- Count one rectangle then multiply by four?
  5x4=20

Do you understand what you are doing in context?



$$1x^{1} + 1x^{1} + 1x^{1} + 1x^{1}$$

$$4x^{1}$$

$$4(5)^{1}$$

$$4(5)$$

$$20$$

If I make x=5, then how many square units do I have?

How many in each line, what is the area of a single rectangle?

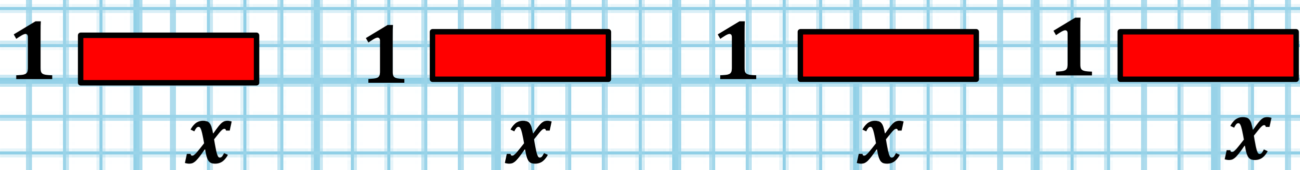
$1 \cdot (5)^{1} = 5$

How many altogether?

$4(5) = 20$

Notice order of operations makes sense in this area context.

What is the area of your four lines/rectangles?



$$4x^{1}$$

$$4(\quad)^{1}$$

$$4(\quad)$$

Find the area of <u>your</u> four lines by substituting into the algebraic expression.

Write it on the other, blank side of your whiteboard.

Show your work using proper notation like that on the left.

Have someone nearby check your work.

Why do Algebra Tiles in Middle School Mathematics?

6.EE.A.1

Write and evaluate numerical expressions involving whole-number exponents.

6.G.A

Solve real-world and mathematical problems involving area, surface area, and volume.

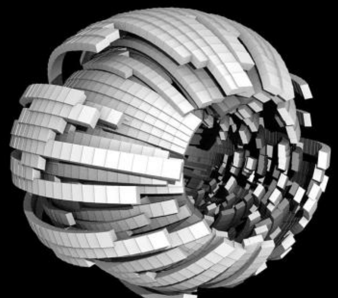
Sixth graders are taught about linear expressions (see next); however, they are just thrown into quadratic and cubic terms with no understanding of what they are or what they mean.

This is a shame as area, surface area & volume are sixth grade standards as well.

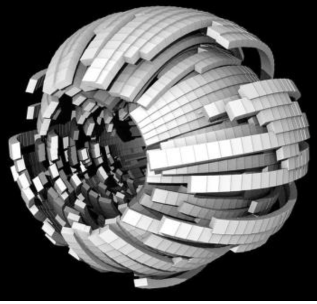
Algebra teachers, wouldn't you love students coming in with this knowledge?

Levels of Abstraction aka Concreteness Fading

David Mattoon
Meaning for Memory



Levels of Abstraction = Concreteness Fading
Concrete, Representational, Abstract



During the first half of my teaching career, I would spend what seemed to be the first half of a math lesson teaching a new math concept by sharing definitions, formulas, steps and procedures.

To make things more challenging for my students, I would simultaneously introduce the symbolic notation used to represent those ideas. Then, I would spend the **remainder** of the lesson **attempting** to help my students make sense of these very **new** and often **abstract ideas**.

By the end of the lesson, I could help many students build an understanding, but there was **always a group I felt who I would leave behind.** Like many other teachers, I was just **teaching in a very similar way to that how I was taught**. **I knew no different.**

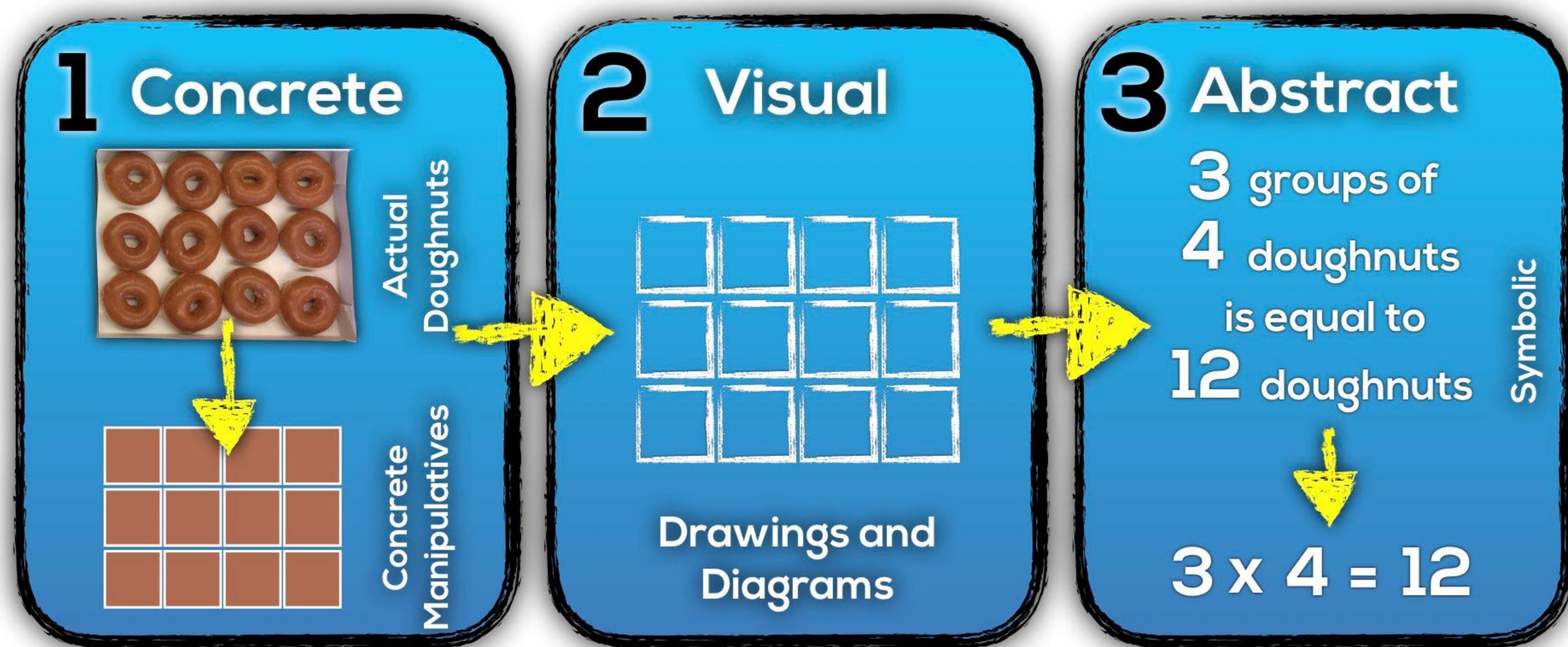
However, if we consider that new learning requires the linking of new information with information they already know and understand, we should be intentionally planning our lessons with this in mind. **A great place to start new learning is through the use of a meaningful context and utilizing concrete manipulatives that students can touch and feel.**

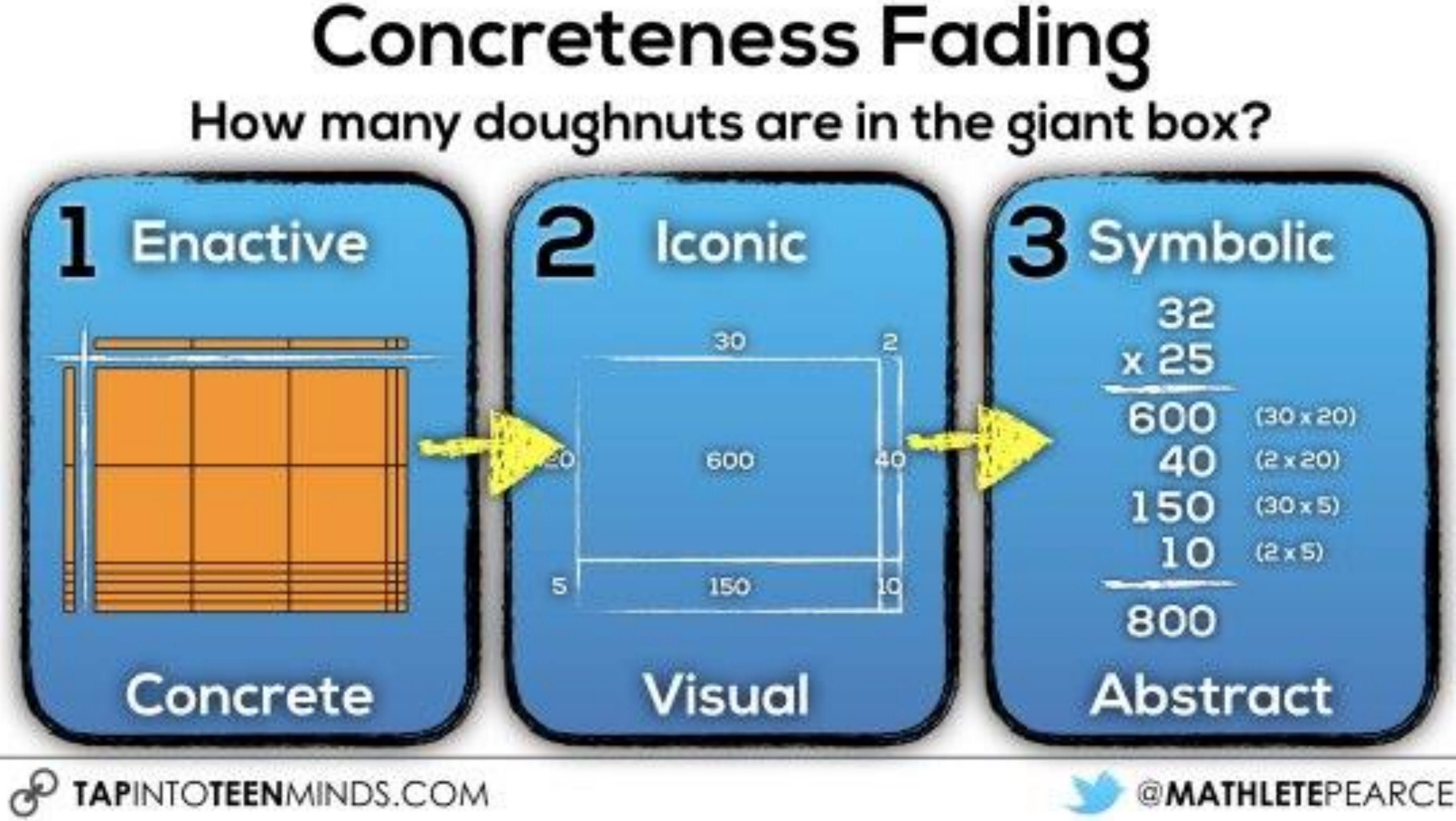
When we teach in this way, we **minimize the level of abstraction** so students can **focus their working memory on the new idea** being introduced in a **meaningful way.**

- Kyle Pierce, Tap into Teen Minds, https://tapintoteenminds.com/concreteness-fading/

Concreteness Fading

How many donuts are in 4 boxes of 12 donuts?



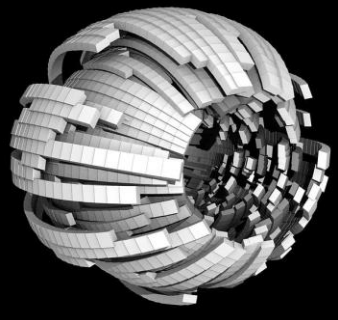


Jerome Bruner (1966) proposed three modes of representation:
Enactive representation (action-based)
Iconic representation (image-based)
Symbolic representation (language-based)

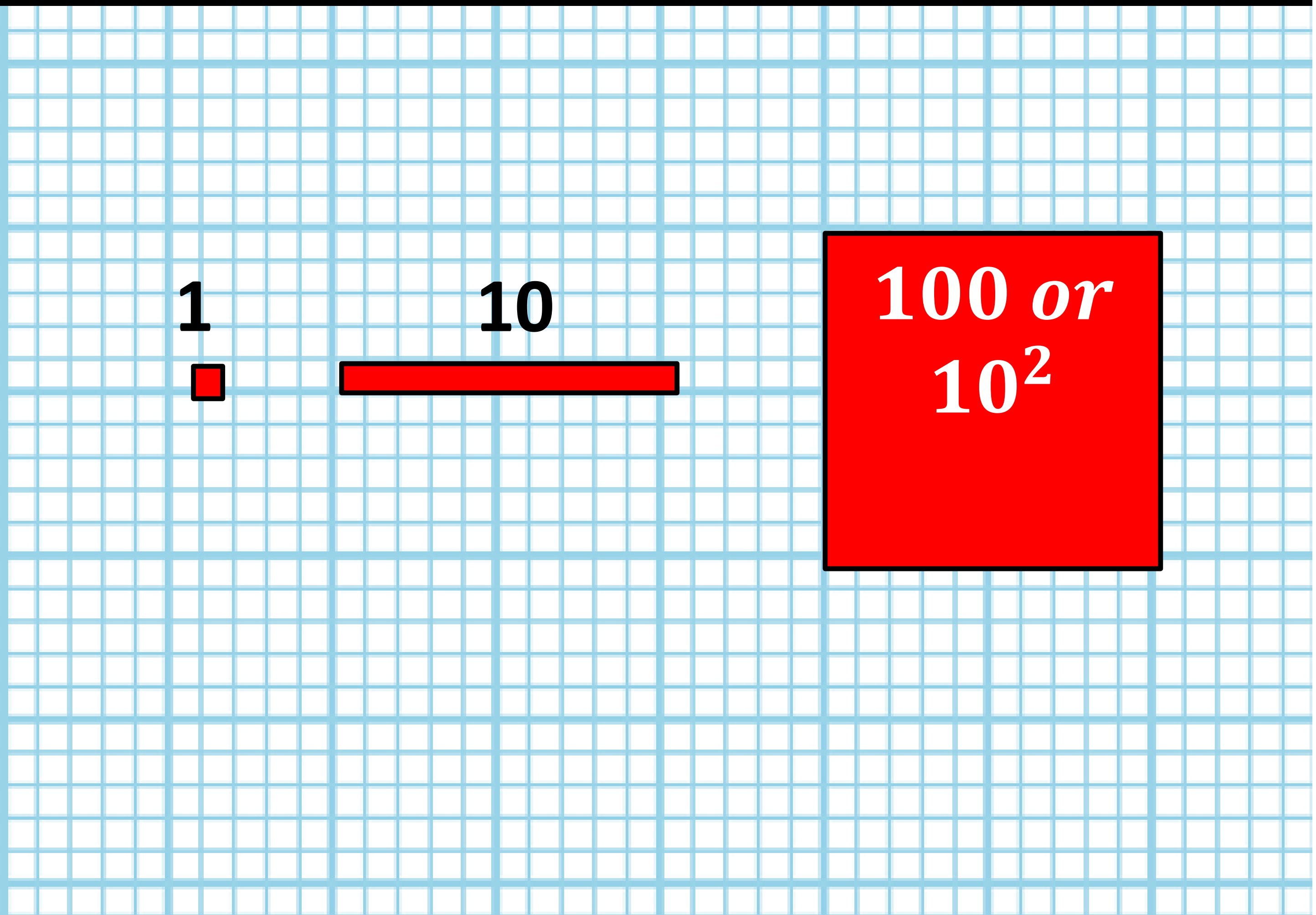
Polynomial Expressions: What do they mean?

David Mattoon

Meaning for Memory



Base Ten Blocks

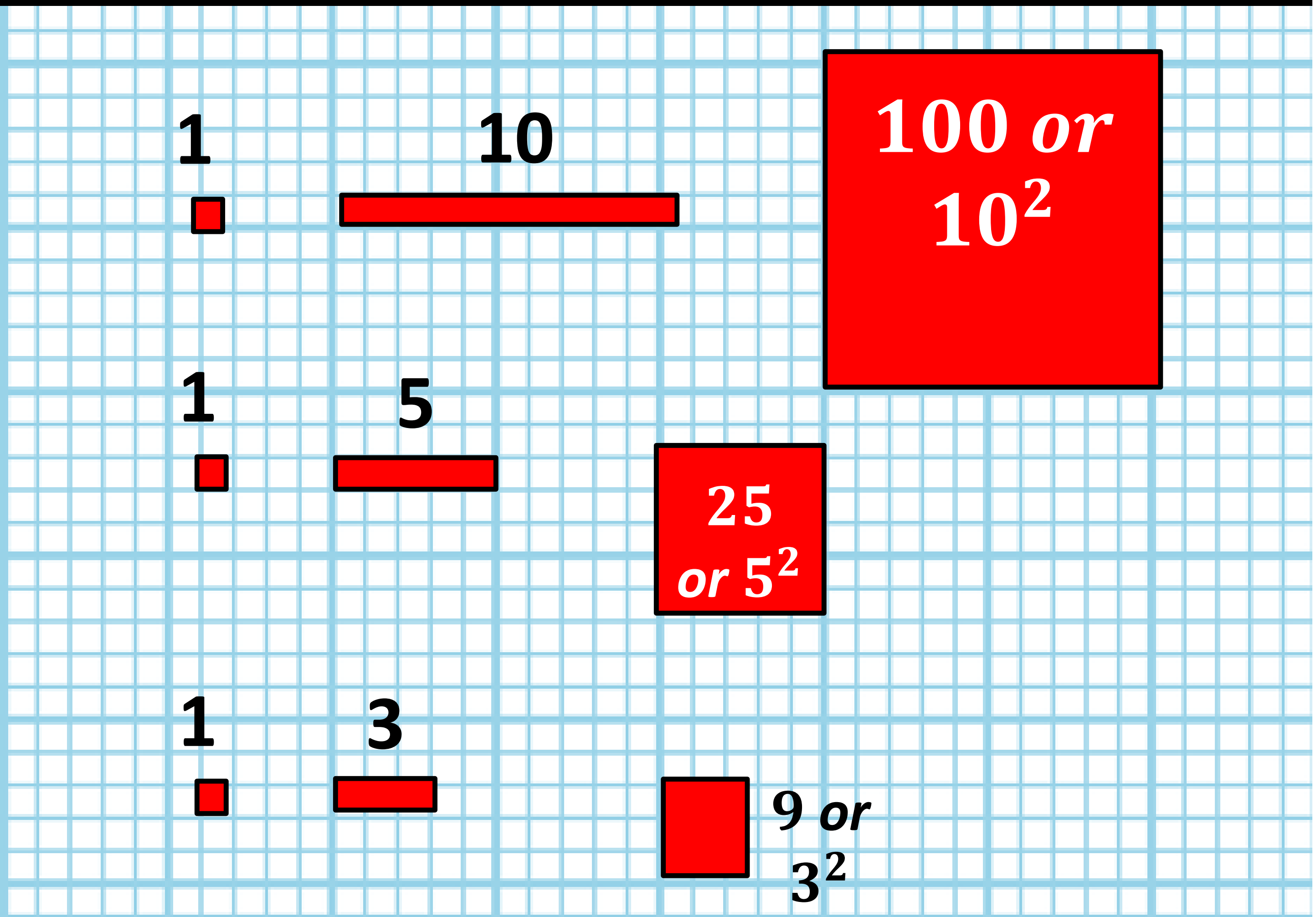


You have to define what one is. When doing decimals, the quadratic term is one.

In the case of whole numbers, one is the small square, which makes the others ten and one hundred.

Begin referring to 100 and “ten squared” or ten in two dimensions, length and width.

What does the base have to do with it?



Numbers in different bases; one of each tile in each base.

In base 10, one of each is the number 111.

In base 5, one of each is the number 31.

In base 3, one of each is the number 13.

How many people had something like this in their classroom growing up?

It should have been something like this:

Place Value Chart in Base Ten

1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
1000.	100.	10.	1.	0.1	0.01	0.001

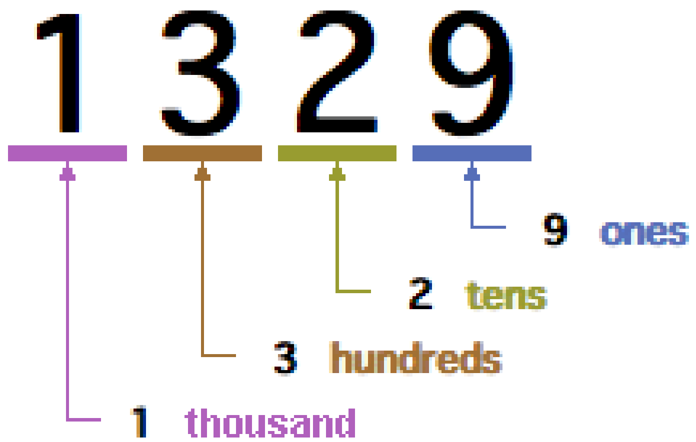
Top row: multiples of tens

Middle row: powers of ten (just names for multiples of ten)

Bottom row: decimal names for multiples of ten

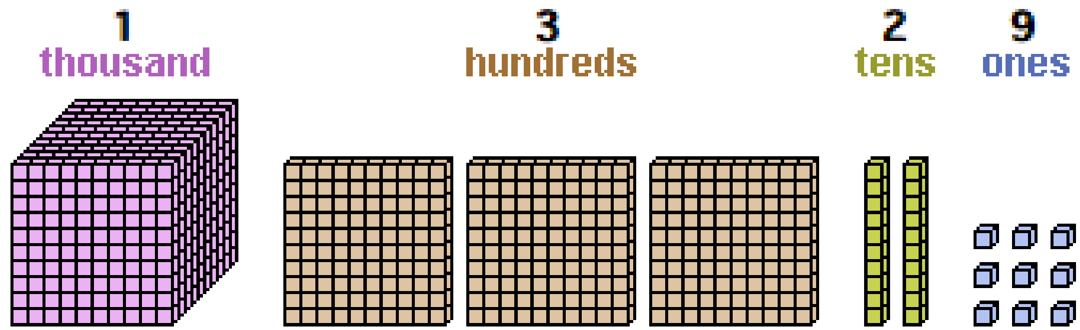
8

Or This:

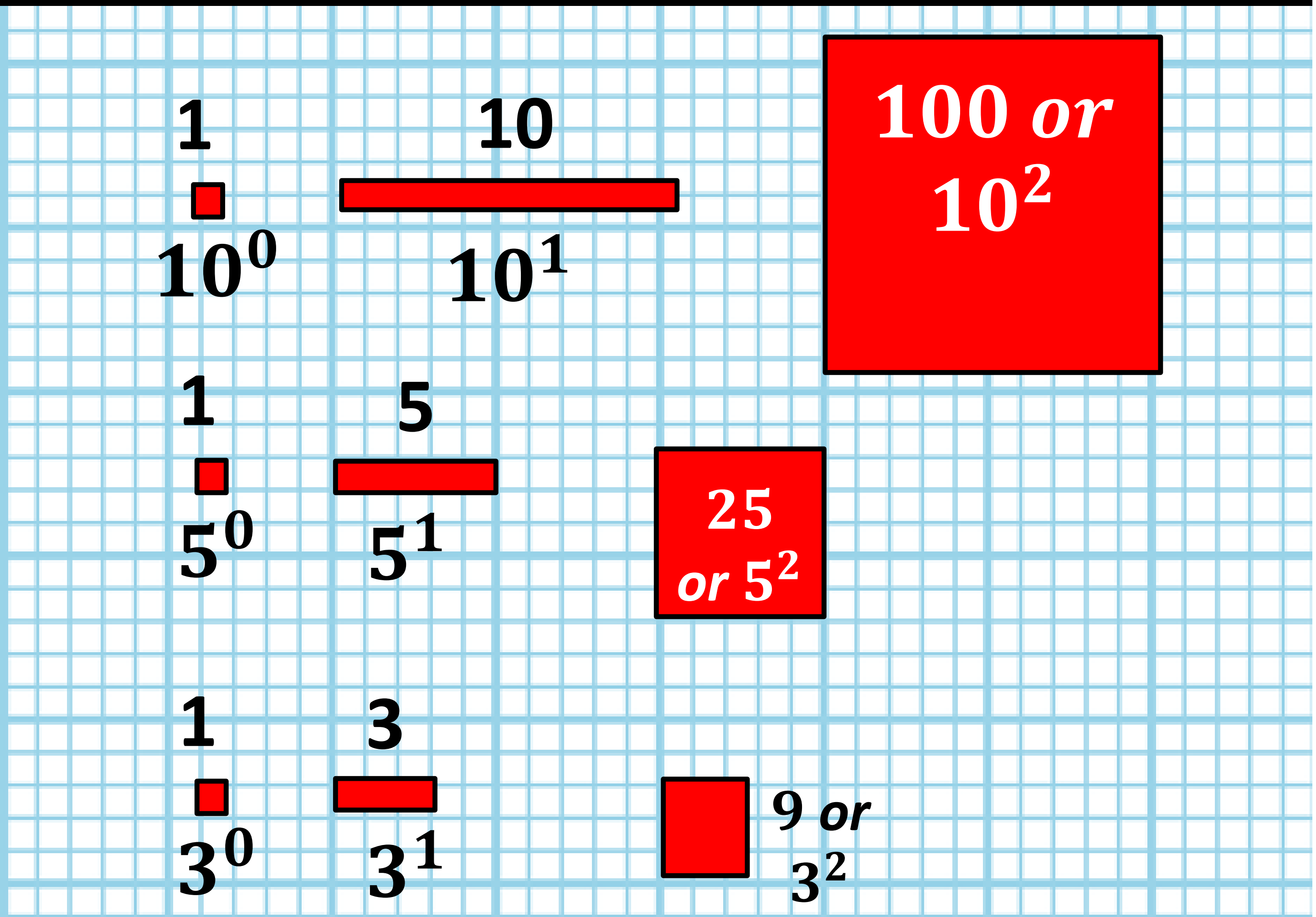


1329 : one thousand, three hundred and twenty-nine

$$1329 = (1 \times 1000) + (3 \times 100) + (2 \times 10) + (9 \times 1)$$

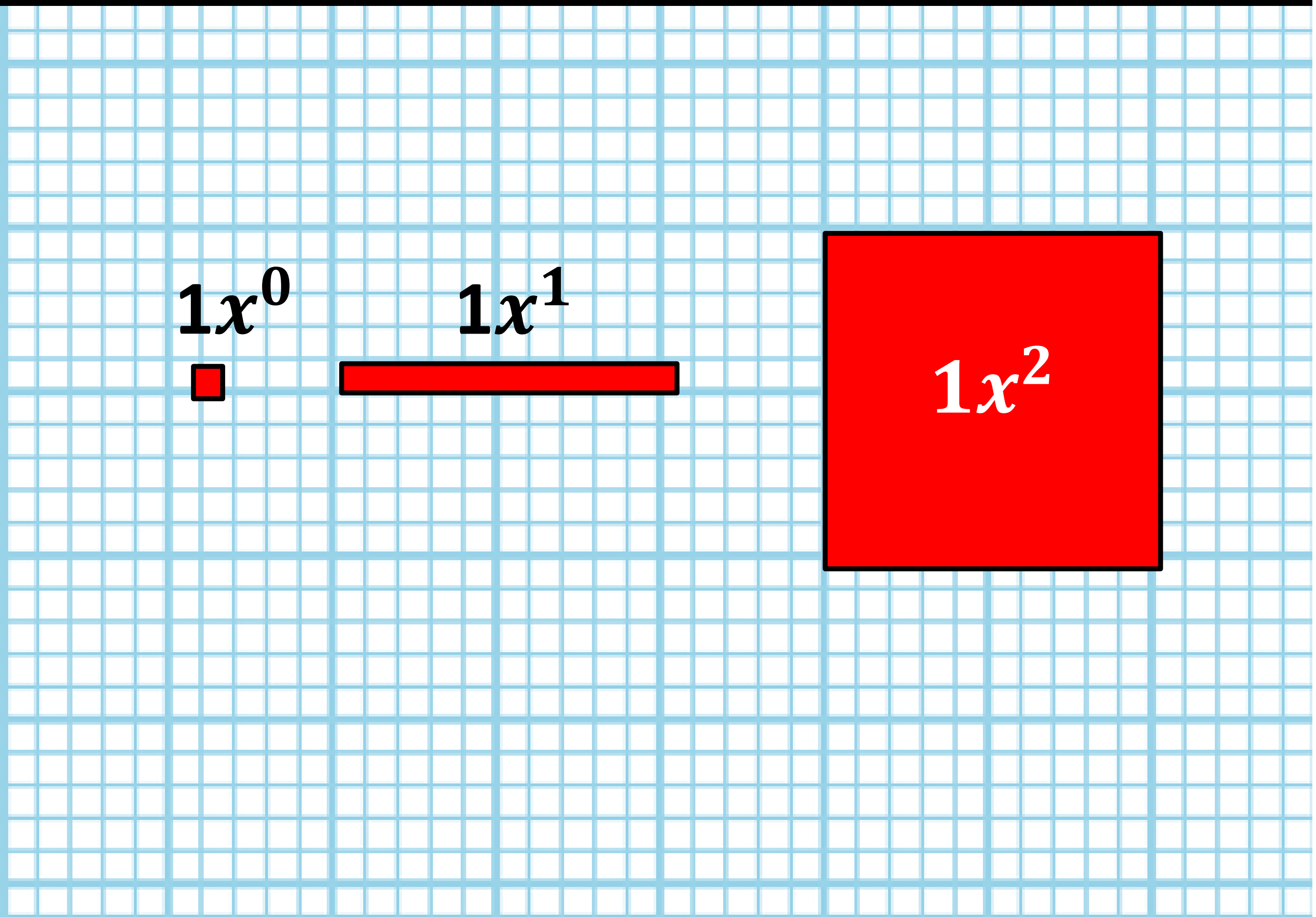


What Varies in Each Expression?



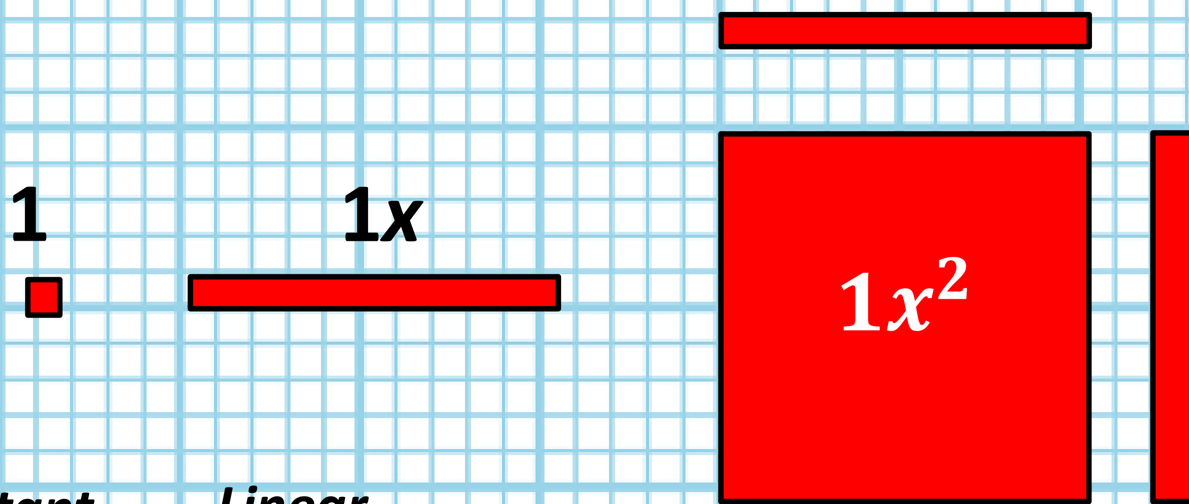
What if it was any base?

Algebra Tiles



Our number system is based on powers of ten; therefore, it is the exponent, which drives what shape it is.

Algebra Tiles



Constant –

a length of 1 and a height of 1

Linear –

a line of length x and a height of 1

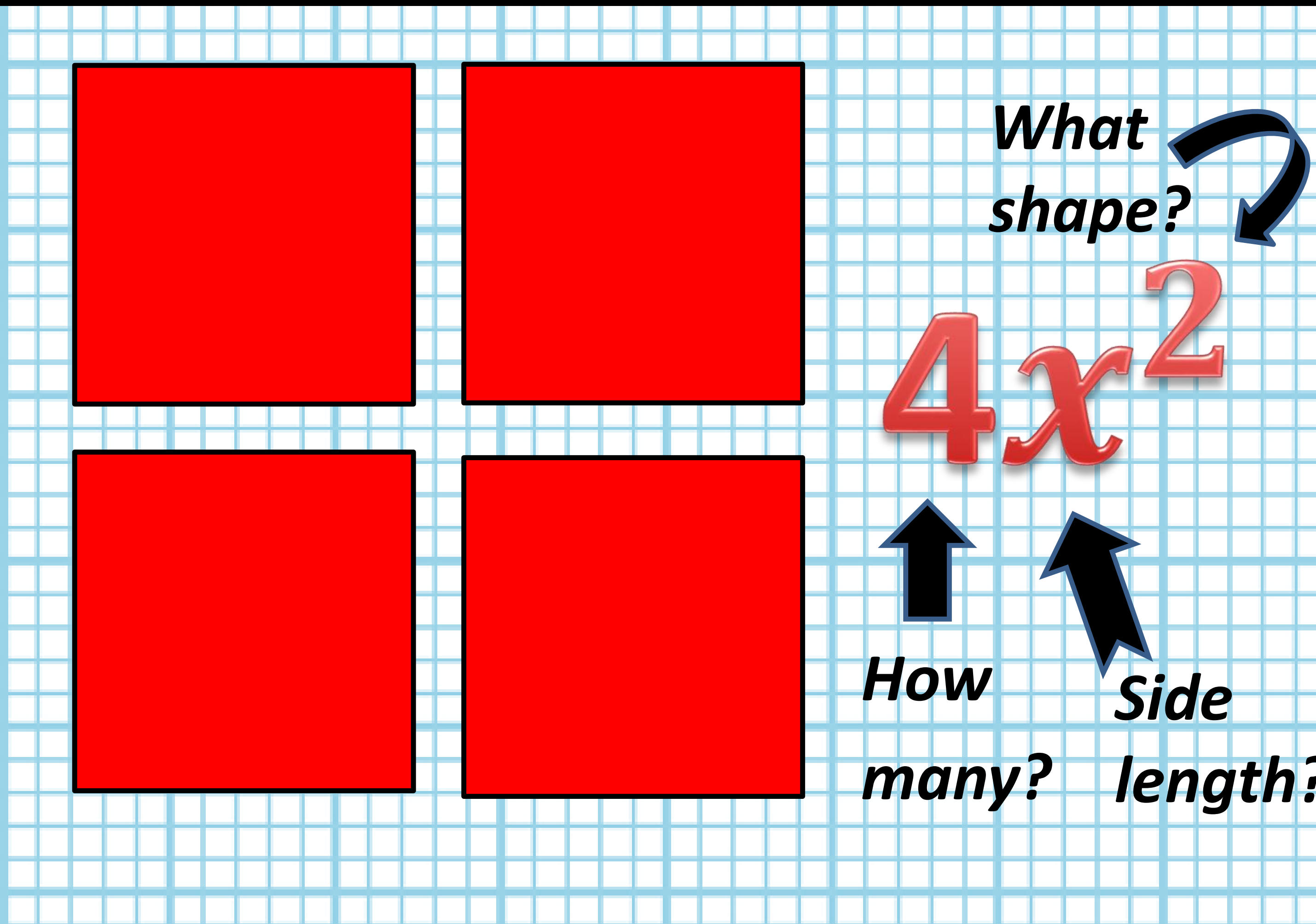
Quadratic–

a square with a side length of x.

In the case of algebra tiles, one is the small square, which is the same as base ten. You might say it is constantly one. Really it is a constant , because the value of x will not influence it.

Begin referring to x squared as a square with a side length of x. A square with two dimensions, a length of x and a width of x.

Anatomy of an Algebraic Term



You want students to discover this and/or be able to tell you when you ask:

What does the exponent mean?

What does the base mean?

What does the coefficient mean?

Combining Polynomials: Addition

David Mattoon

Meaning for Memory

Algebra Tiles: *Show Me*

($1x^2 + 2x + 3$)

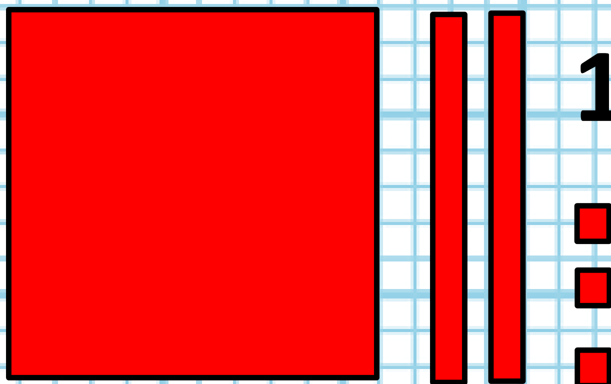
$$1x^2 + 2x + 3$$

Use the blank side of your whiteboard as your work space.

The unused tiles are off the whiteboard.

Algebra Tiles: Addition

$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$



$1x^2 + 2x + 3$

$2x^2 + 3x + 4$

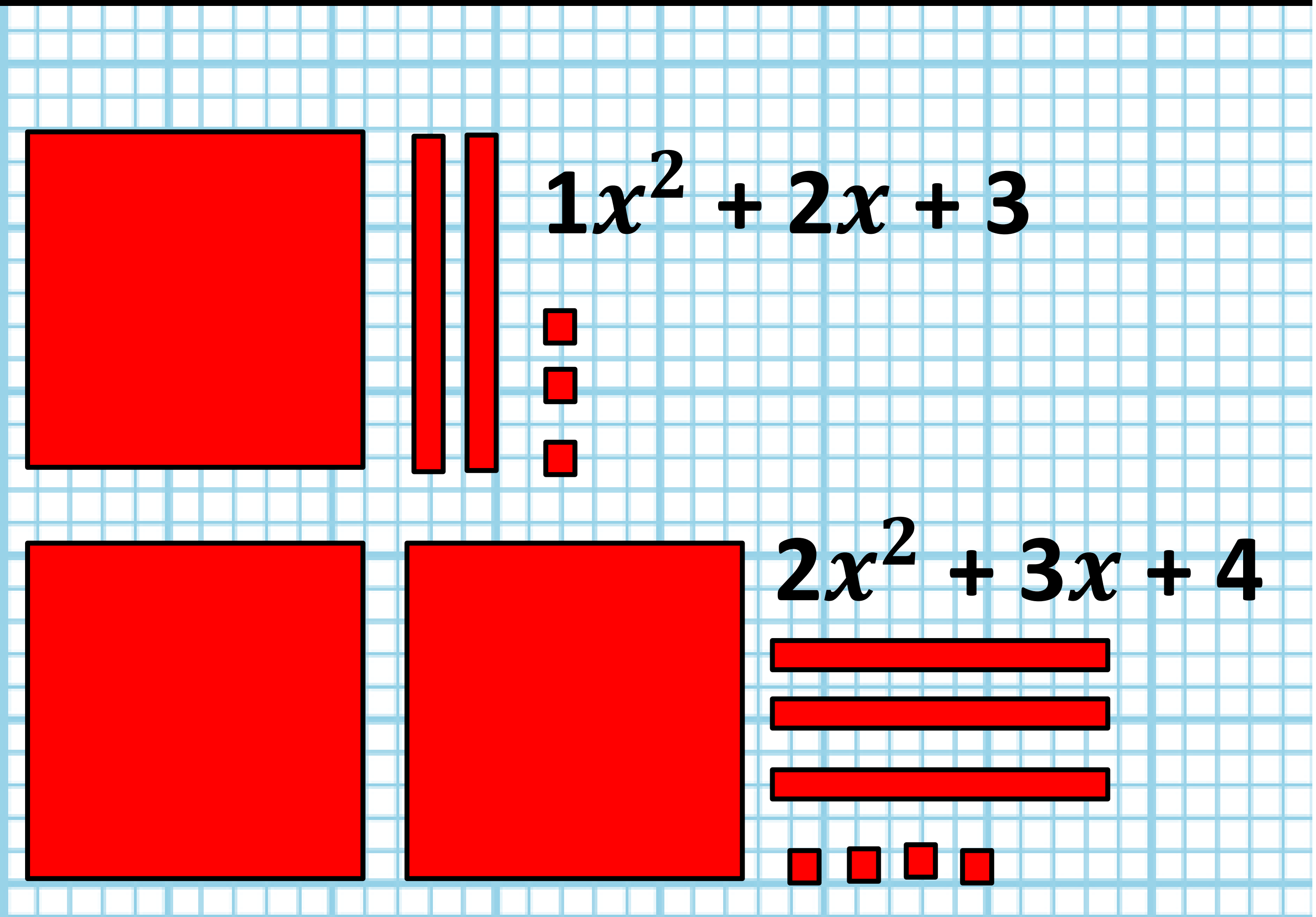
Use the blank side of your whiteboard as your work space.

Add the polynomials

The sum is on the whiteboard, and the unused tiles are off the whiteboard.

Algebra Tiles: Addition

$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$

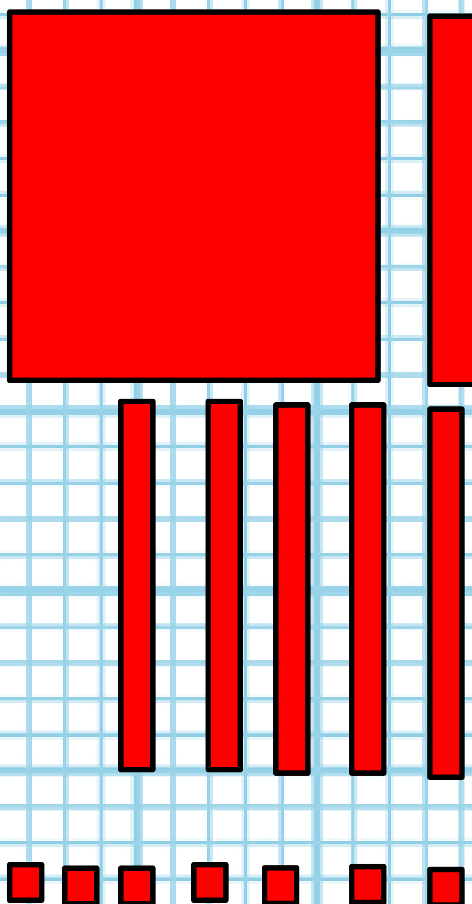


Each expression individually

By the way, does it matter how you draw the linear term, vertically or horizontally?

Algebra Tiles: Addition

$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$



$$
\begin{array}{r}
1(x^2) + 2(x) + 3(1) \\
+\ 2(x^2) + 3(x) + 4(1) \\
\hline
3(x^2) + 5(x) + 7(1)
\end{array}
$$

$$\boxed{3x^2}\ \boxed{+\ 5x} + 7$$

Have them put the same shapes together, aka combine like terms.

Can you put the terms together? No, they are not the same shape. Squares, Lines & Dots or Quadratic Terms, Linear Terms & Constants

Cognitive Demand & Access

"My students aren't ready for algebra."

"They have to master basic skills first before learning algebra."

While it is preferred to see algebra as a generalization of any base, could we use algebra to remediate number? Instead of going specific to general could we go general to specific?

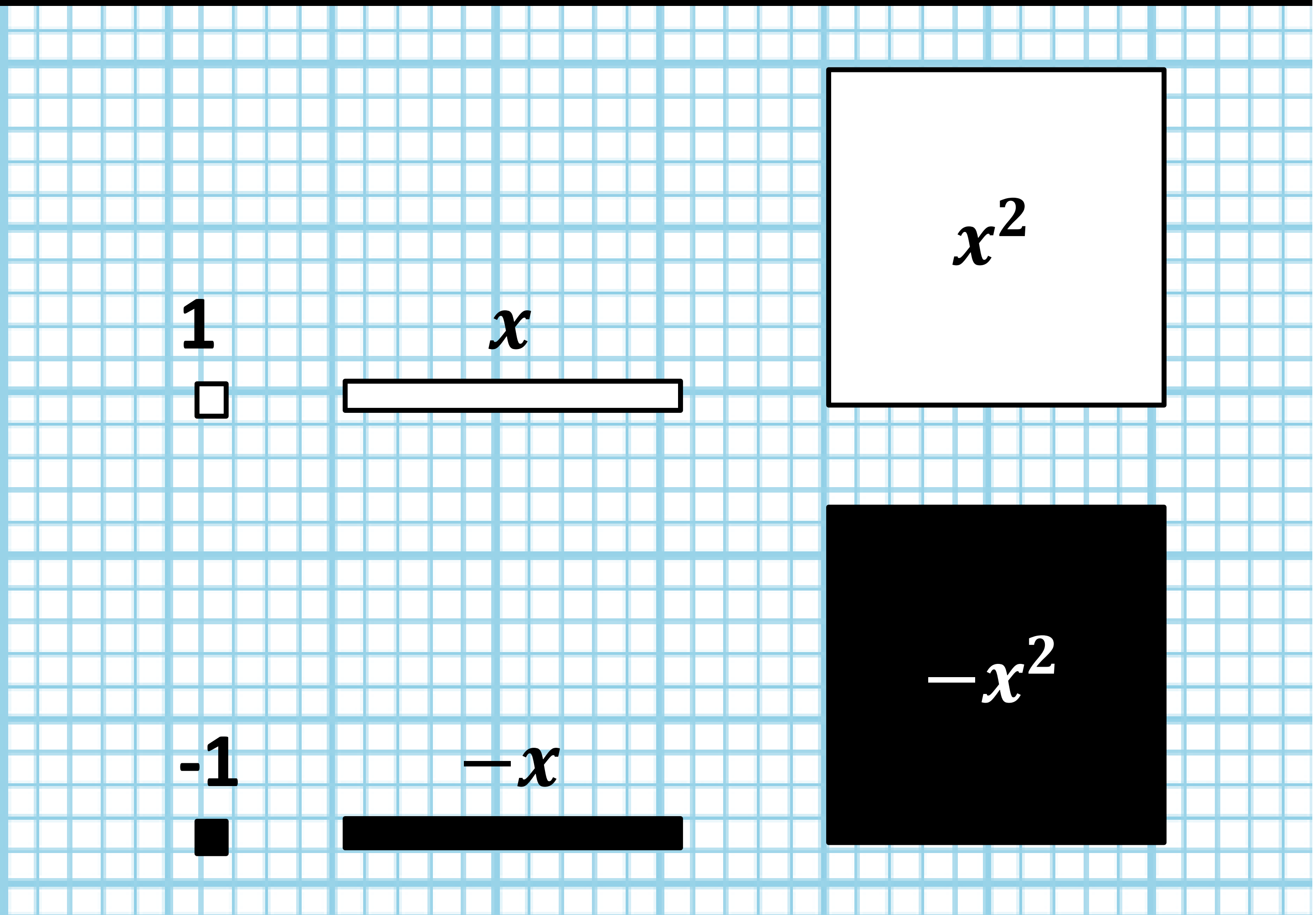
Keep students on grade level and remediate number with Tier Two interventions using more time to relate number to algebra. In class, use polynomial addition & subtraction to remediate integers or even whole numbers.

Remediate in context of grade level material.

There is no regrouping with algebra tiles as you do not know what the base is, or how many units it takes to make the units of the next power.

The lack of regrouping actually makes the cognitive demand for learning algebra less than the cognitive demand for number when using algebra tiles and base ten blocks.

Next Level of Abstraction: Drawing the Tiles Under the Expressions



"Open" is positive.

"Closed" is negative.

Referring to them as white being positive and black being negative is not a good idea.

Fold A Blank Paper into Three Columns & Label

+ =

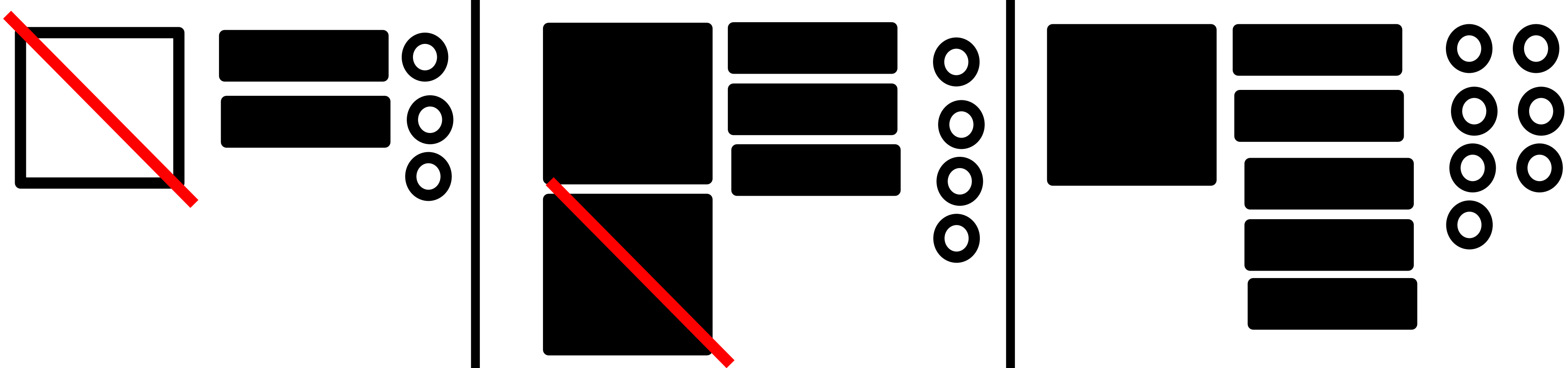
Combining Expressions: Three Column Notes

$1x^2$ -2x + 3	+ -2x^2 -3x + 4	=

Draw the shapes below the expressions, cross out any zero pairs, and write the sum pictorially and symbolically.

Combining Expressions: Three Column Notes

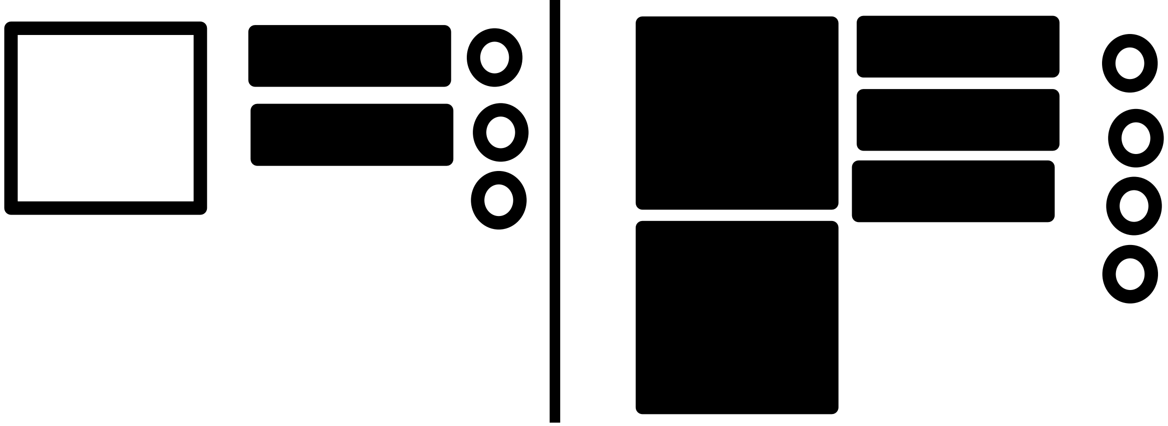
$1x^2$ $-2x + 3$	+ $-2x^2$ $-3x + 4$	= $-1x^2$ $-5x$ $+7$



Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

$$1x^2 - 2x + 3 \quad + \quad -2x^2 - 3x + 4 \quad =$$

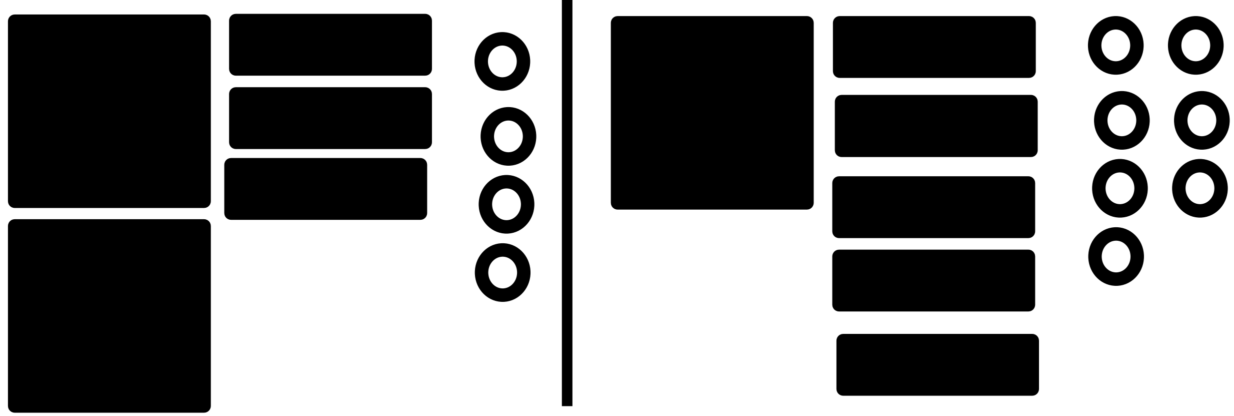


Eliminate just enough information to make it doable.

Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

$1x^2 - 2x + 3$	$+$	$=$ $-1x^2 - 5x + 7$



Eliminate just enough information to make it doable.

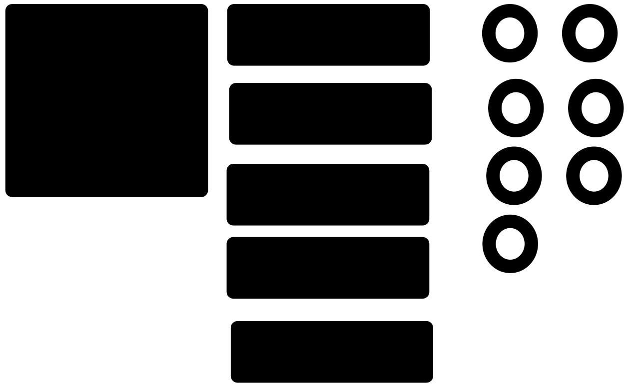
Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

+

=

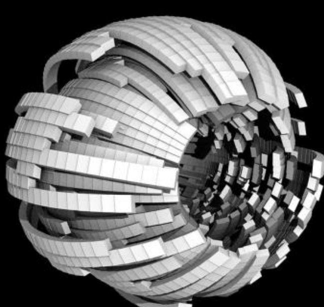
$-1x^2 -5x +7$



Students make their own problem.

Require students to draw the shapes AND write the symbols.

Combining Polynomials: Subtraction



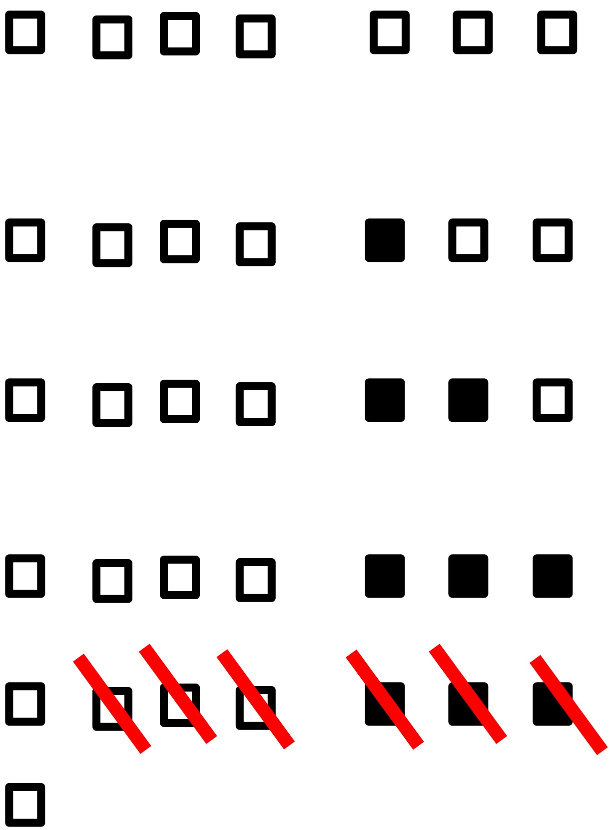
David Mattoon

Meaning for Memory

Integer Subtraction

$(4) - (3) =$

$(4) - (3) = 1$

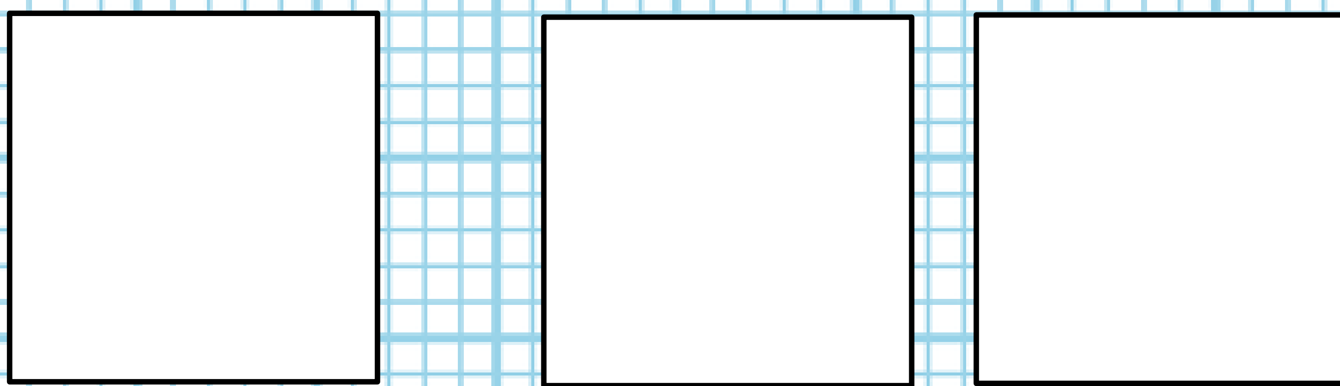


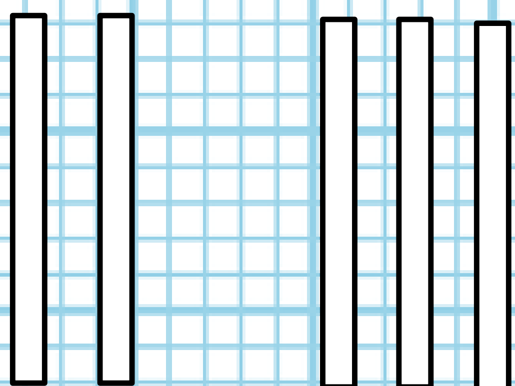
T: What is 4-3?
Ss: 1
T: So, why does this look like 7?
Ss: You have to change them to negatives.
T: Like this?
Ss: No, change all of them!
T: Like this?
Ss: No, no. Change ALL of them!
T: Oh, you mean I have to change ALL of them? All of minuend or subtrahend?
Ss: All of the subtrahend.

After integers, my students are familiar with zero pairs. Before subtracting polynomials, I give them this problem.

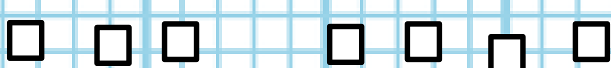
Subtracting Expressions with Algebra Tiles

$(1x^2 + 2x + 3) - (2x^2 + 3x + 4)$





$$\begin{array}{r} (1x^2 + 2x + 3) \\ -\,(2x^2 + 3x + 4) \\ \hline \end{array}$$



THIS IS NOT CORRECT!

Negative means opposite so which do we switch, the minuend or the subtrahend?

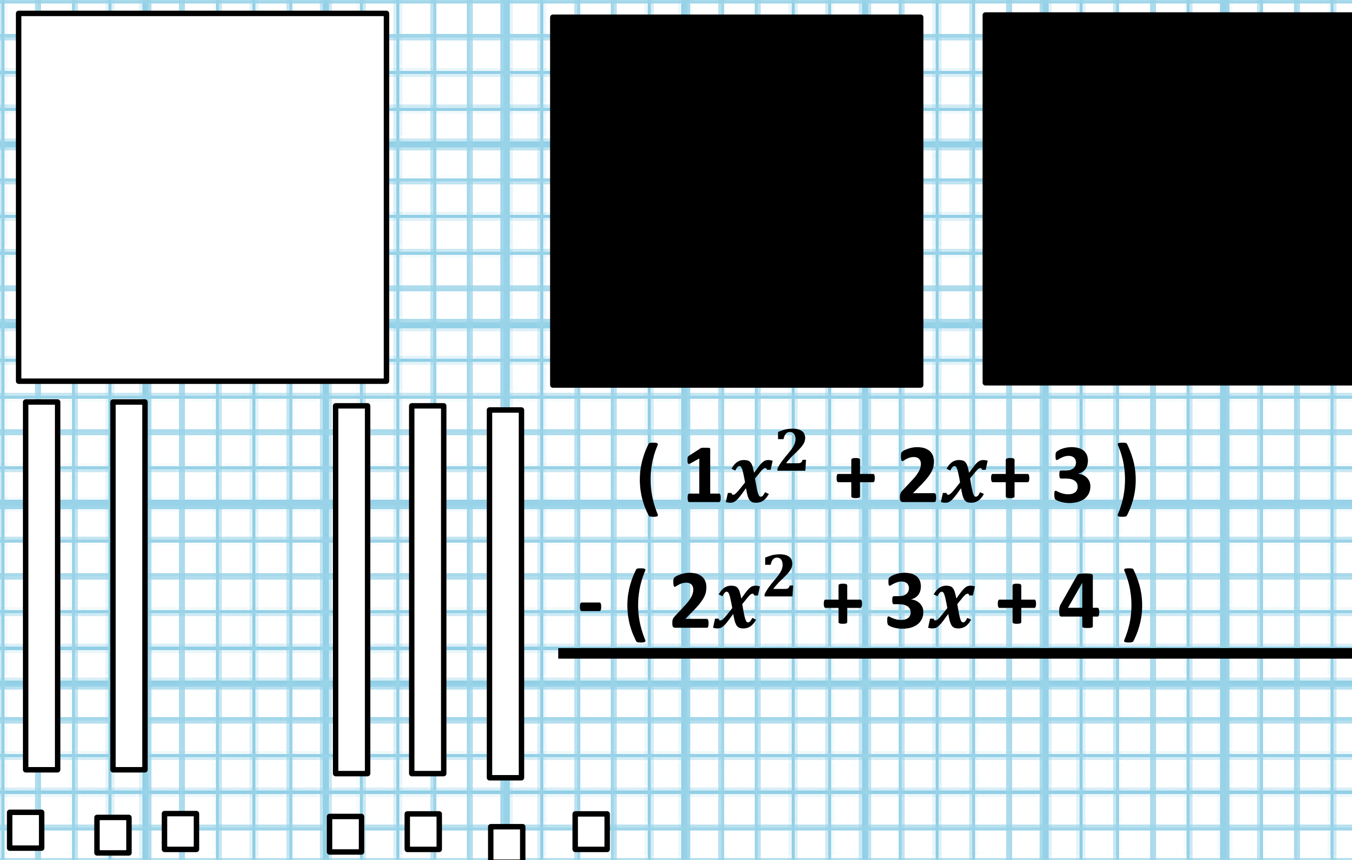
Which did we switch with 4-3?

Do we change one of them or all of them?

At some point, you can point out this is the equivalent to distributing a negative one.

Subtracting Expressions with Algebra Tiles

$$(1x^2 + 2x + 3) - (2x^2 + 3x + 4)$$



$$\begin{array}{r} (1x^2 + 2x + 3) \\ -\ (2x^2 + 3x + 4) \\ \hline \end{array}$$

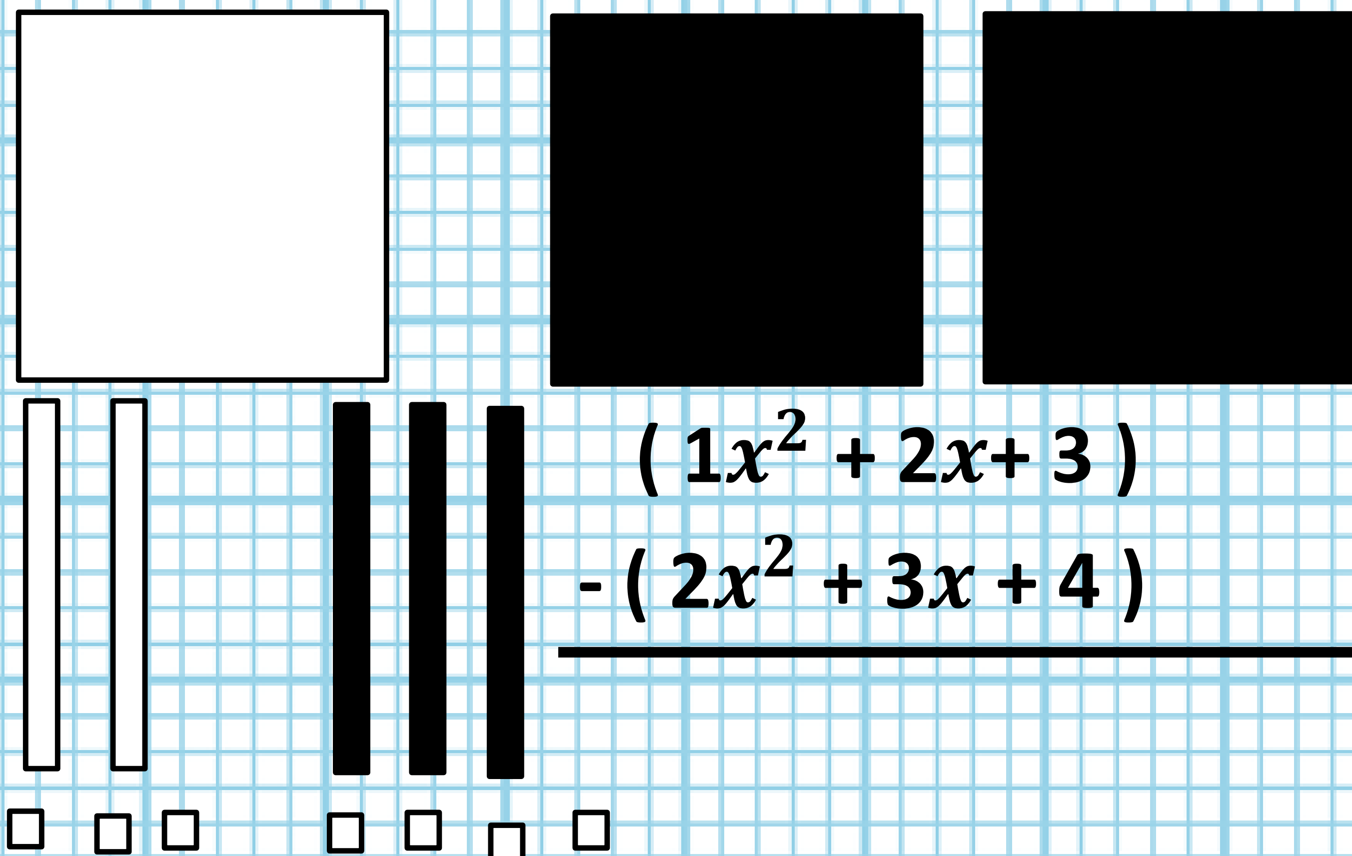
THIS IS NOT CORRECT YET

What is the opposite of $2x^2$?

$-2x^2$ so shade it in.

Subtracting Expressions with Algebra Tiles

$$(1x^2 + 2x + 3) - (2x^2 + 3x + 4)$$



$$\begin{array}{r} (1x^2 + 2x + 3) \\ -\,(2x^2 + 3x + 4) \\ \hline \end{array}$$

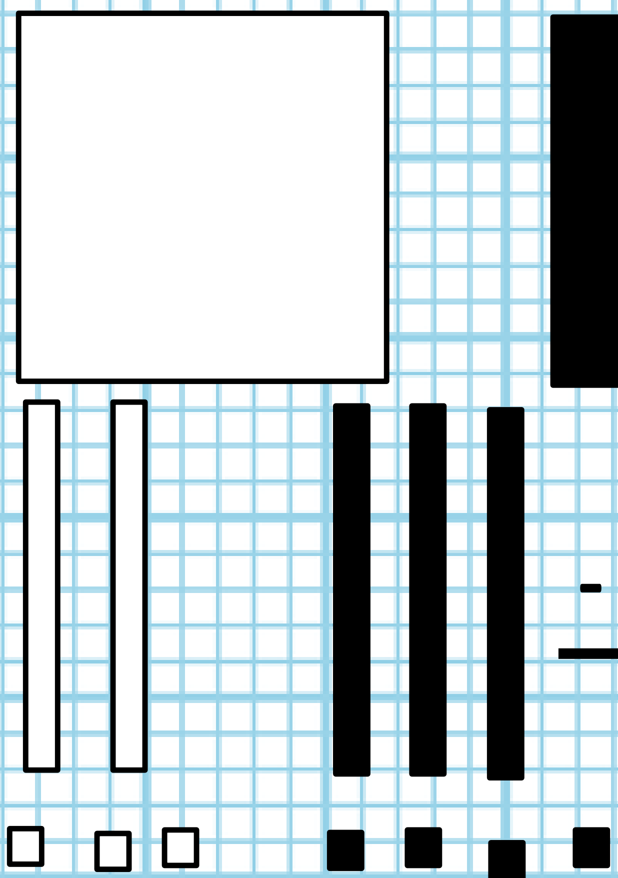
THIS IS NOT CORRECT YET

What is the opposite of $+ 3x$?

$- 3x$ so shade it in.

Subtracting Expressions with Algebra Tiles

$$(1x^2 + 2x + 3\ ) - (2x^2 + 3x + 4)$$



$$\begin{array}{r} (\ 1x^2 + 2x + 3\ ) \\ -\ (\ 2x^2 + 3x + 4\ ) \\ \hline \end{array}$$

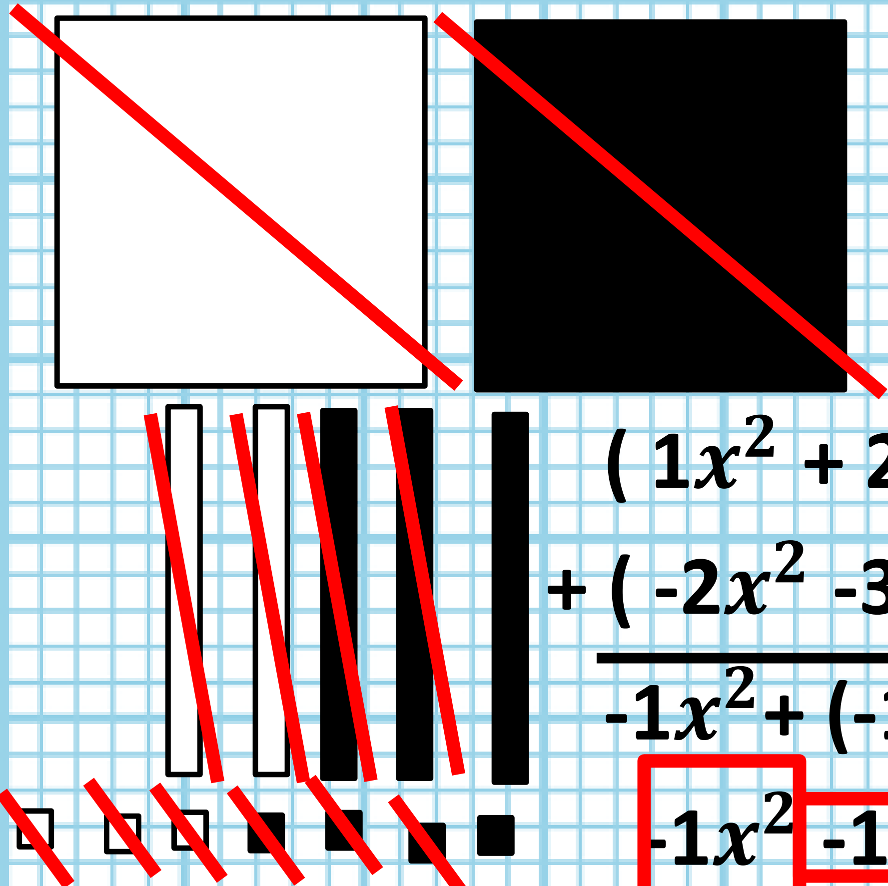
THIS IS NOT CORRECT YET

What is the opposite of + 4?

- 4 so shade it in.

Subtracting Expressions with Algebra Tiles

$$(1x^2 + 2x + 3) + (-2x^2 - 3x - 4)$$



$$
\begin{array}{r}
(1x^2 + 2x + 3) \\
+\ (-2x^2 - 3x - 4) \\
\hline
-1x^2 + (-1x) + (-1)
\end{array}
$$

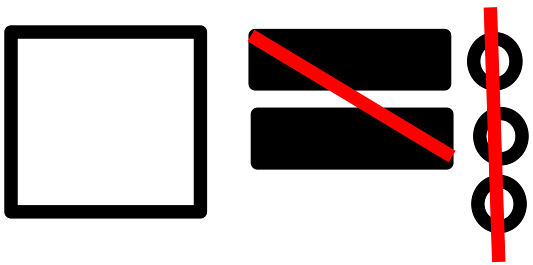
$$\boxed{-1x^2}\ \boxed{-1x}\ -1$$

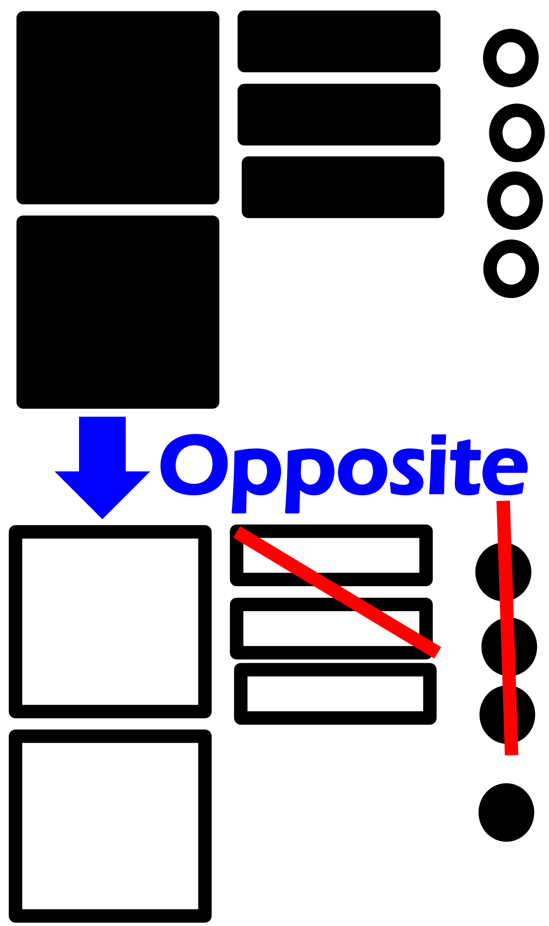
From our work with integers using a context of hot rocks and cold rocks, students know that subtracting is the same as adding the opposite.

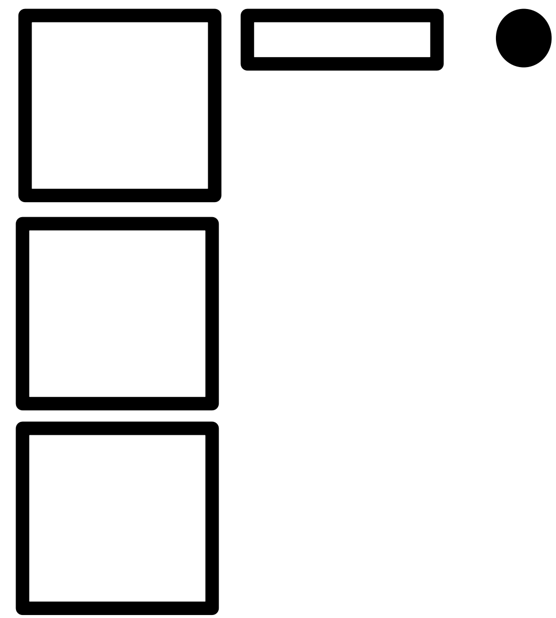
You can raise the temperature of a drink by either removing an ice cube or adding a hot rock.

Combining Expressions: Three Column Notes

$(1x^2 - 2x + 3) - (-2x^2 - 3x + 4) = 3x^2 + 1x - 1$



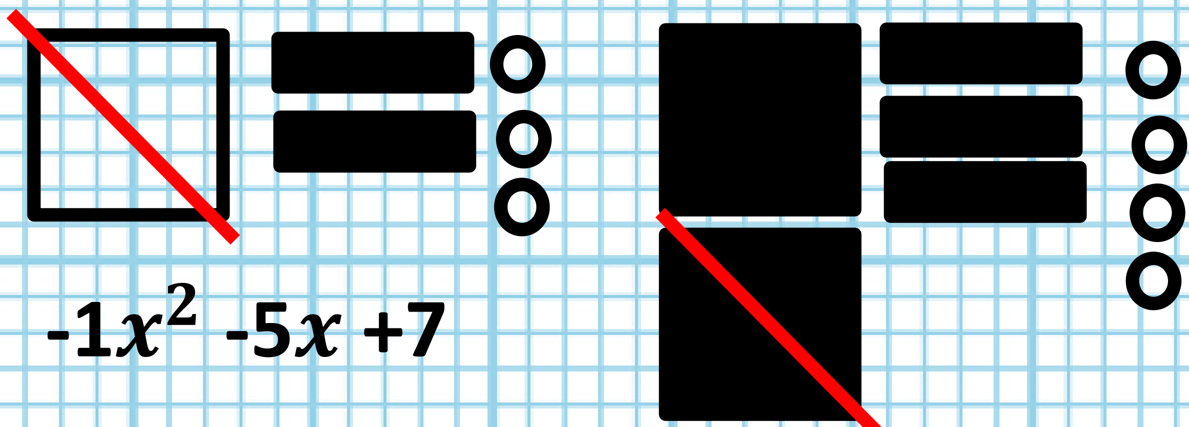




If you do subtraction, then make them change the shapes to the opposite before combining.

Next Level of Abstraction: Drawing the Tiles Around the Symbols

$(1x^2 \ -2x \ +3) + (-2x^2 \ -3x \ +4)$



$-1x^2 \ -5x \ +7$

$(11x^2 +32x +13) + (22x^2 -34x +14)$

$33x^2 \ -2x \ +27$

This is a good time to introduce larger coefficients.

THE HORIZONTAL METHOD IS GOOD FOR DRAWING SHAPES AROUND OR BELOW.

Make your squares, quadratic, large and your rectangles, linear, smaller, but with the same side length.

Constants get circles instead of the little squares that are more difficult to draw and easily confused with quadratics.

Final Level of Abstraction: Line Up Terms Vertically

($1x^2$ $-2x$ $+3$) - ($-2x^2$ $-3x$ $+ 4$)

$$
\begin{array}{rrr}
+2x^2 & +3x & -4 \\
\hline
3x^2 & +1x & -1
\end{array}
$$

Keep asking, "Can I simplify?"

THE VERTICAL METHOD IS GOOD FOR KEEPING LIKE TERMS TOGETHER AND REWRITING SUBTRACTION.

You can reference the shapes as you do it . You want students to follow the procedure of combining like terms based on what they learned about the areas they represent.

You can redraw them at the end to reinforce it.

Final Level of Abstraction: Line Up Terms Vertically

$$(\,1x^2 \;\; -2x \;\; +3\,) - (\,-2x^2 \;\; -3x \;\; +4\,)$$

$$\begin{array}{rrr} +2x^2 & +3x & -4 \\ \hline \boxed{3x^2} & \boxed{+1x} & -1 \end{array}$$

No, quadratic, linear & constant

THE VERTICAL METHOD IS GOOD FOR KEEPING LIKE TERMS TOGETHER AND REWRITING SUBTRACTION.

You can reference the shapes as you do it . You want students to follow the procedure of combining like terms based on what they learned about the areas they represent.

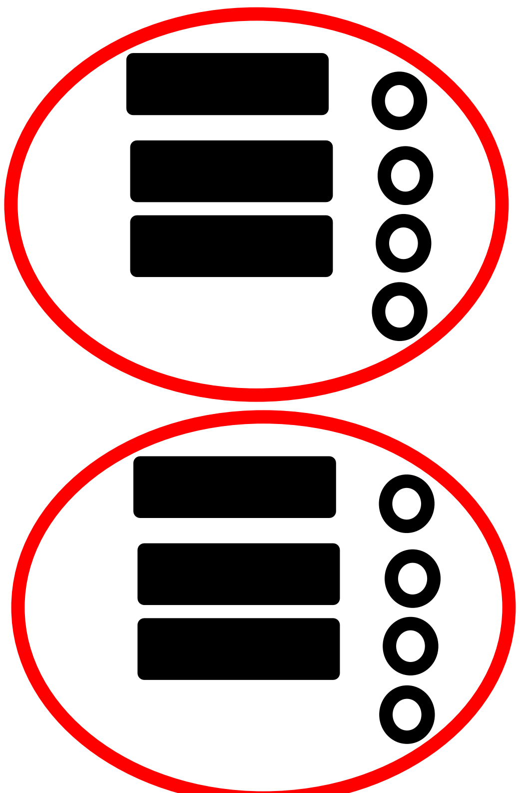
You can redraw them at the end to reinforce it.

The Distributive Property

David Mattoon
Meaning for Memory

Distributive Property: Reading & Drawing the Tiles

2 (-3x+ 4) Read, “Two groups of negative three x plus four or two groups of negative three times a number plus four.” Drawn as:



One group

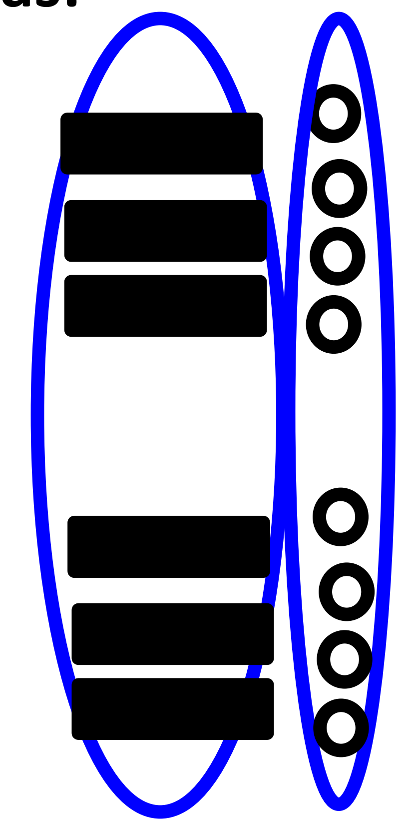
Two groups

How many negative x’s?

How many constants?

So, the product is?

$6x + 4$



Most students haven’t had the conceptual instruction for multiplication.

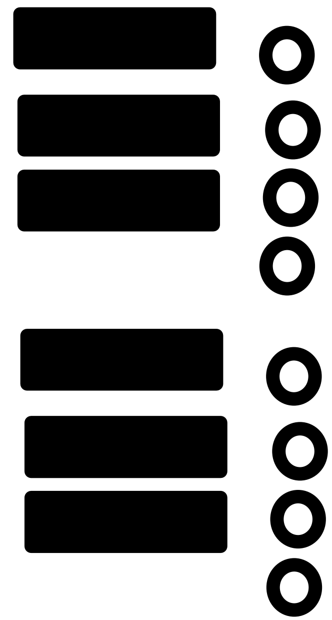
Most can multiply, but some do not realize it is making groups.

Others can tell you it is making groups, but it still does not inform their mathematics.

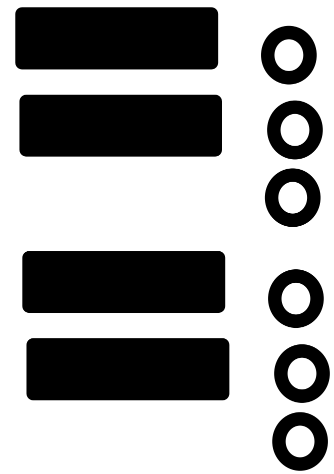
For example, they don’t make groups to simplify.

This needs to be reinforced when distributing.

Distributive Property: Four Column Notes

Symbolic	Meaning in Words	Pictorial	Simplified
$2(-3x+4)$	Two groups of negative three times a number plus four		$-6x+8$
	Three groups of two times a number minus five		

Distributive Property: Four Column Notes

Symbolic	Meaning in Words	Pictorial	Simplified
			
$-2(4x - 3)$		Opposite	

Substitution & Context for Linear Expressions

David Mattoon
Meaning for Memory

Context as Support

I went through the drive-thru, and I received two hamburgers and $4 change. Write an algebraic expression for what happened using the variable h.

What does h mean in this context?

Context as Support

$$2h + 4$$

What does h mean in this context?

If two is the number of hamburgers, then h must be some other <u>number</u>. h means more than hamburger. It could be the weight of the meat or the price of the burger, but it has to be some <u>number</u>.

In this context, four being an amount of money, it is the price of the burger.

Substitution into a Linear Expression

$$2h + 4$$

I paid the drive-thru employee with a single bill. What was the single bill I paid with if hamburgers cost three dollars or $h = \$3$?

Substitution into a Linear Expression

$$2h + 4$$

A paid the drive-thru employee with a single bill. What was the single bill I paid with if hamburgers cost three dollars or $h = \$3$?

$$2(3) + 4$$

$$6+4$$

$$10$$

It was a ten dollar bill.

How does Order of Operations make sense in this context?

Linear Substitution: Give Peas a Chance!

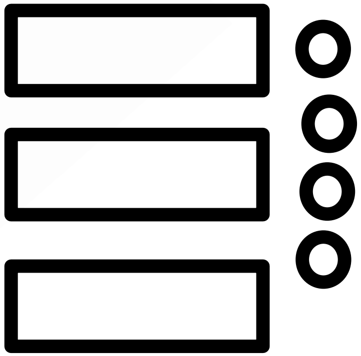
California Mathematics Council
Southern Section

Another Context for Linear Substitution

- How are the peas arranged in a pod?
- You can think of a pea pod as a linear term.
- Let $\square$ represent the peapod.
- Let $\circ$ represent the peas

If p= # of peas in a pod, then…

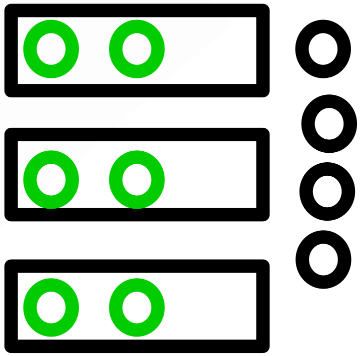
Another Context for Linear Substitution



If p= # of peas in a pod, then..

How many total peas if the pea pods hold two peas each?

Another Context for Linear Substitution



If p= # of peas in a pod, then..

How many total peas if the pea pods hold two peas each?

How did you do it?

$3p + 4$

$3(2) + 4$

$6 + 4$

10

How does Order of Operations make sense in this context?

Another Context for Substitution

On your whiteboard, draw the expression using

□ to represent the peapod.

○ to represent the peas

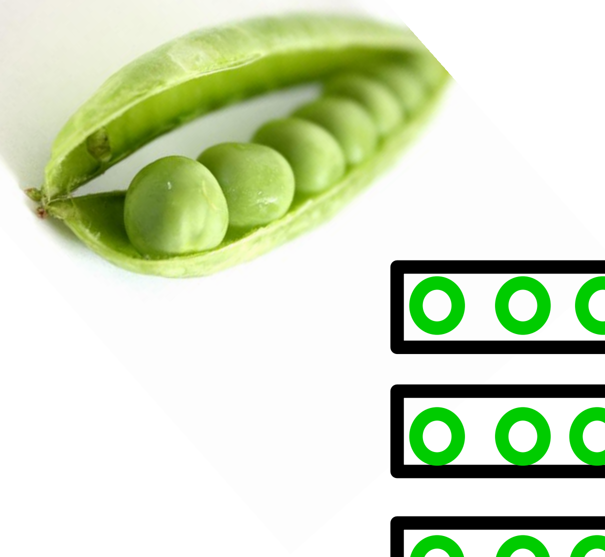
Show the substitution of the **three peas** in your picture.

If p= # of peas in a pod, then..

How many total peas if the pea pods hold three peas each?

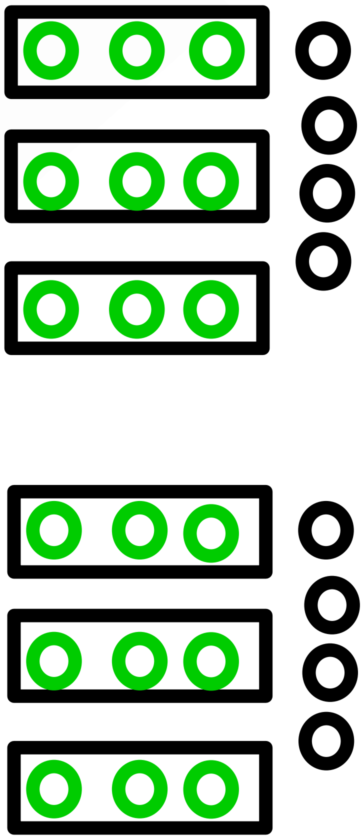
$2(3p + 4)$

Another Context for Substitution



If p= # of peas in a pod, then..

How many total peas if the pea pods hold three peas each?



$2(3p + 4)$

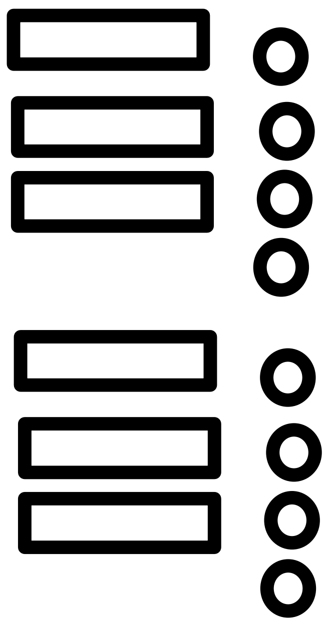
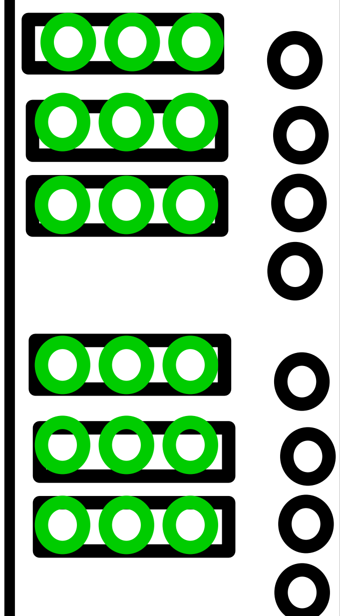
$6p + 8$

$6(3) + 8$

$18 + 8$

26

Fold Your Legal Paper into Five Columns

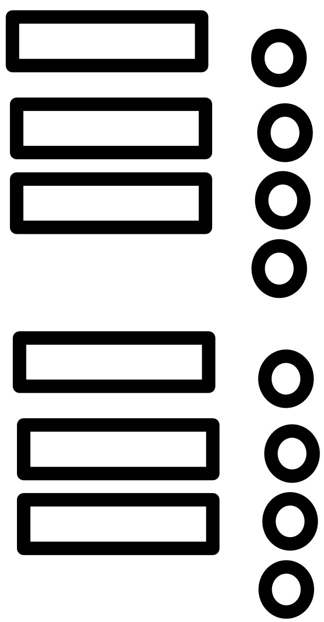
Linguistic in Context of Peas	Pictorial Expression	Symbolic Expression	If $p = 3$ Pictorial	If $p = 3$ Symbolic
Three pea pods and four extra peas, twice		$(3p + 4)2$		$(3p + 4)2$ $[3(3) + 4]2$ $[9 + 4]2$ $[13]2$ 26

Fold Your Legal Paper into Five Columns

Linguistic in Context of Peas	Pictorial Expression	Symbolic Expression	If $p = 3$ Pictorial	If $p = 3$ Symbolic
Three pea pods and four extra peas, twice				

Fold Your Legal Paper into Five Columns

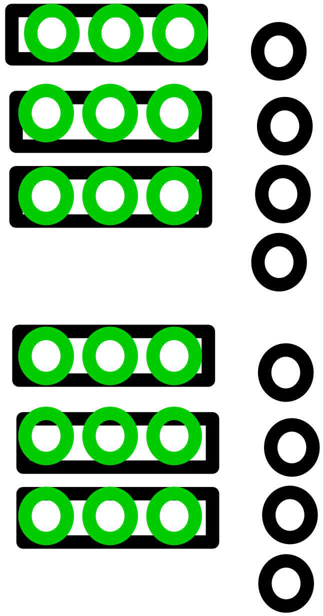
Linguistic in Context of Peas	Pictorial Expression	Symbolic Expression	If $p = 3$ Pictorial	If $p = 3$ Symbolic



Fold Your Legal Paper into Five Columns

Linguistic in Context of Peas	Pictorial Expression	Symbolic Expression	If $p = 3$ Pictorial	If $p = 3$ Symbolic
		$(3p + 4)2$		

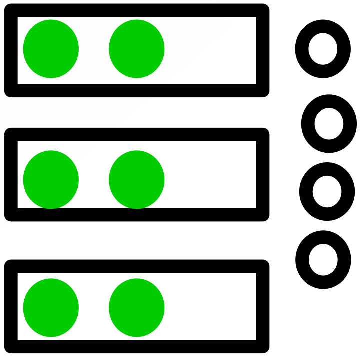
Fold Your Legal Paper into Five Columns

Linguistic in Context of Peas	Pictorial Expression	Symbolic Expression	If $p = 3$ Pictorial	If $p = 3$ Symbolic
				

Fold Your Legal Paper into Five Columns

Linguistic in Context of Peas	Pictorial Expression	Symbolic Expression	If $p = 3$ Pictorial	If $p = 3$ Symbolic
				$[3(3) + 4]2$ $[9 + 4]2$ $[13]2$ 26

Context for Negative Values (not the best, but it works)



Wouldn't it be weird to open a peapod and find a note saying you owe peas?

If p= # of peas in a pod, then..

Would you owe or have peas if the pea pods hold negative two peas each? How many?

$3p + 4$

$3(-2) + 4$

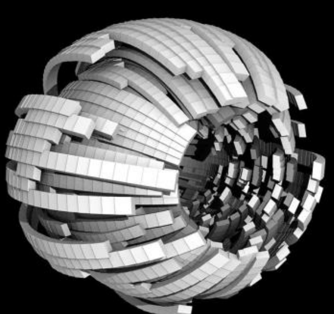
$-6 + 4$

-2, You would owe two peas.

Linear Equations in One Variable

David Mattoon

Meaning for Memory



Equations: Misconceptions

- I have to "scaffold" equations
  1. One step (6th Grade)
  2. Two step (7th Grade)
  3. Distributive
  4. Variables on both sides
  5. Multistep Equations
- Ninth grade algebra starting with 6th grade, one step equations, then 7th grade, etc.
- Worse when compounded by extended beginning of the year review

- Why am I out of time?
- "Those students" get less access to grade level material and the achievement gap widens, because of who?
- Even if students taught procedurally remember for a procedural unit test, what happens when the prompts are not as procedural or the following year when students have forgotten?

If you just finished multistep expressions, then why not start there with equations to reinforce what you have just done? It allows you to get those kids who were almost there all the way there too.

Scaffolding Defined

Instructional scaffolding is the support given during the learning process... to promote a deeper level of learning... These supports may include the following:

- **resources [using pictorial math provides a resource to employ]**
- **a compelling task [not a simplified task]**
- **templates and guides [column notes for multiple representations]**
- **guidance on the development of cognitive skills**
  [How are students thinking about what they are doing?]

These supports are gradually removed as students develop autonomous learning strategies, thus promoting their own cognitive, affective and psychomotor learning skills and knowledge. Teachers help the students master a task or a concept by providing support [not simplifying content]. The support can take many forms such as outlines, recommended documents, storyboards, or key questions.

https://en.wikipedia.org/wiki/Instructional_scaffolding

I went through the drive-thru and gave them $10.

I got back two hamburgers and $4.

How much did the burgers cost?

Context as a Scaffold - Equations

I went through the drive-thru and gave them $10. I got back two hamburgers and $4. How much did the burgers cost?

Students can do this problem without instruction in how to solve an equation, because the context is the scaffold. This refutes the argument that students must start with one step equations before solving two step equations.

Start with the conceptual, see how students do it, and then formalize a procedure based on what they did.

Context as a Scaffold - Equations

I went through the drive-thru and gave them \$10. I got back two hamburgers and \$4. How much did the burgers cost?

What did you do to solve this?

If h= the cost of the burgers, then:

$$2h + 4 = 10$$

$$-4 = -4$$

Why did you do this (consider the context)?

$$2h = 6$$

$$\frac{2}{2}h = \frac{6}{2}$$

Why did you do this (consider the context)?

Context as a Scaffold - Equations

I went through the drive-thru and gave them \$10. I got back two hamburgers and \$4. How much did the burgers cost?

What did you do to solve this?

If h= the cost of the burgers, then:

$$2h + 4 = 10$$

$$-4 = -4$$

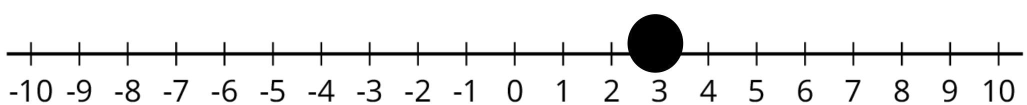
Why did you do this (consider the context)?

$$2h = 6$$

$$\frac{2}{2}h = \frac{6}{2}$$

Why did you do this (consider the context)?

$$h = 3$$

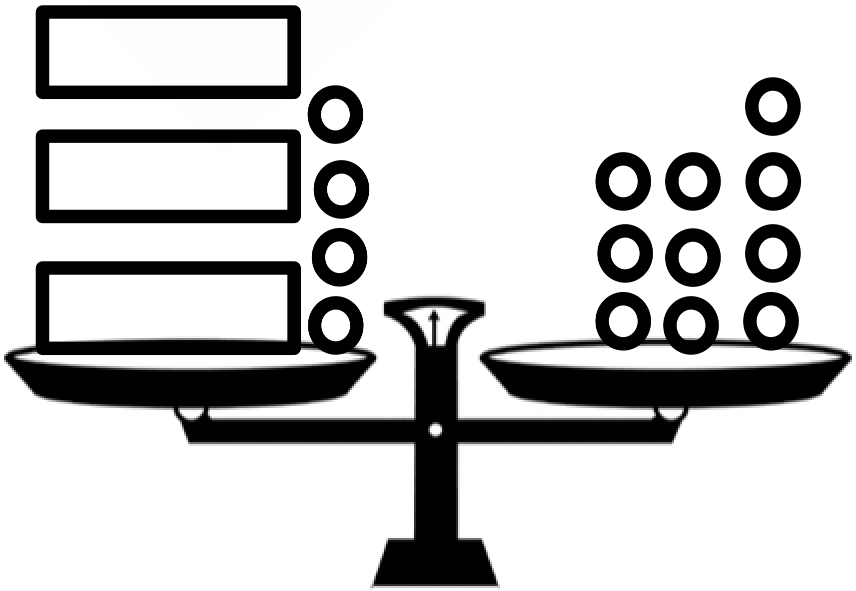


Linear Equations: How Many Peas in the Pod?

Context for Equations

If p= # of peas in a pod, then..

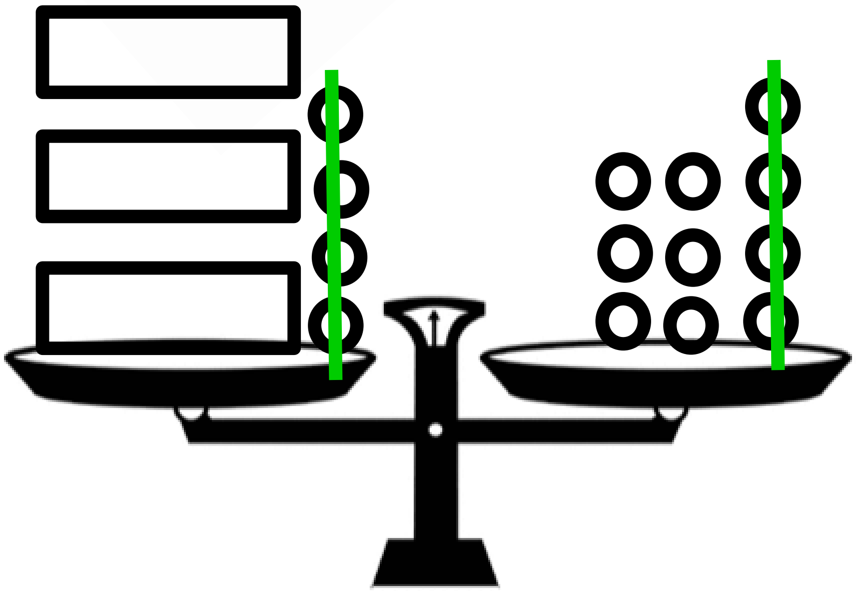
How many peas in the each pea pod? (Disregard the weight of the pod.)



How did you do it?

$$3p + 4 = 10$$

Context for Equations



If p= # of peas in a pod, then..

How many peas in the each pea pod? (Disregard the weight of the pod.)

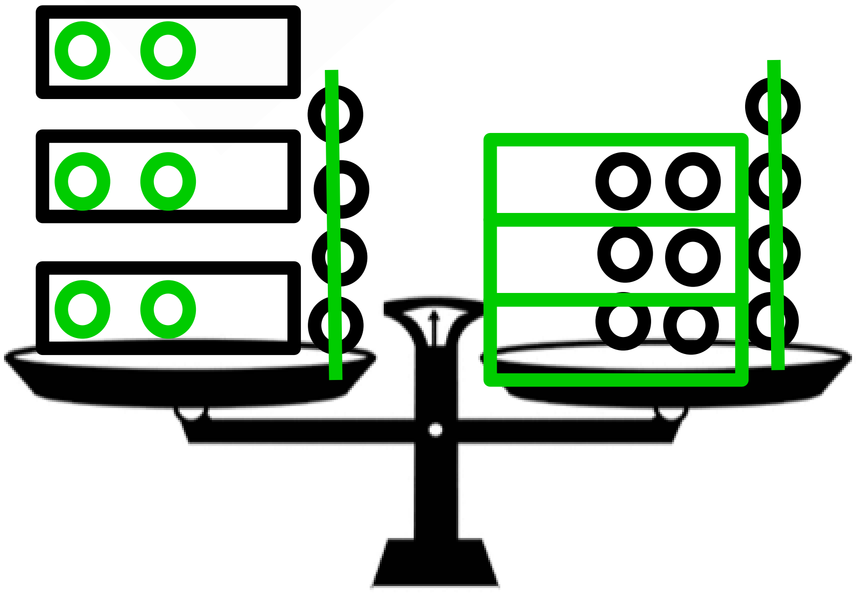
How did you do it?

$$3p + 4 = 10$$

$$3p + 4 - 4 = 10 - 4$$

If p= # of peas in a pod, then..

How many peas in the each pea pod? (Disregard the weight of the pod.)



How did you do it?

$$3p + 4 = 10$$

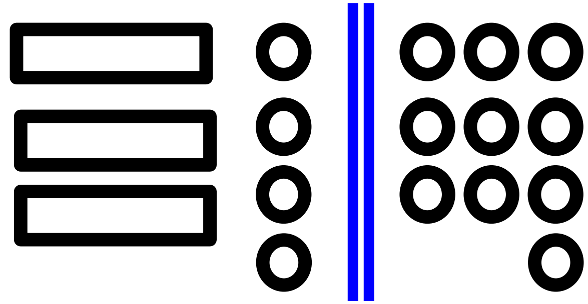
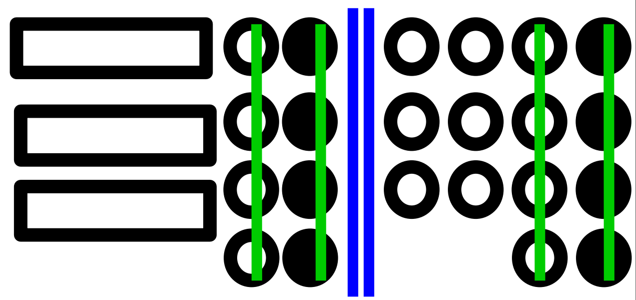
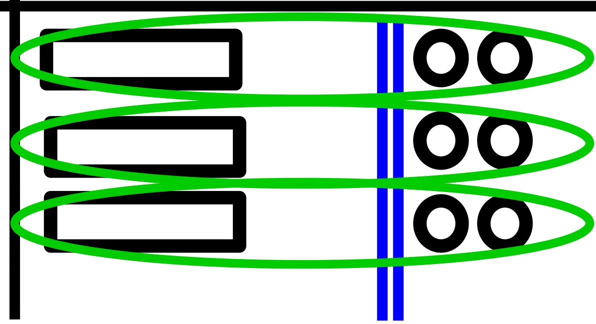
$$3p + 4 - 4 = 10 - 4$$

$$3p = 6$$

$$3/3\ p = 6/3$$

$$p = 2$$

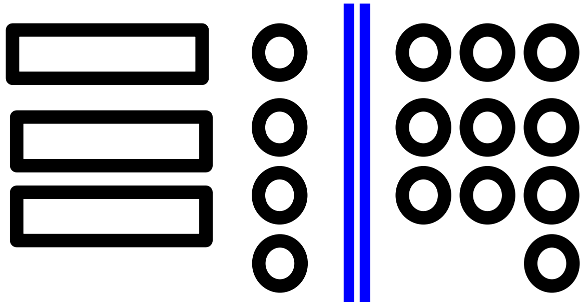
Linear Equations: Four Column Notes

Symbolic	Pictorial	What You Did	Property
$3p + 4 = 10$		Given	Given
$3p + 4 = 10$ $-4 = -4$		Subtracting four from both sides is the same as adding a negative four to both sides.	Addition Property of Equality
$3p = 6$ $3/3p = 6/3$ $p = 2$		I divided both sides by three.	Division Property of Equality

Linear Equations: Four Column Worksheet

Symbolic	Pictorial	What You Did	Property
$3p + 4 = 10$			

Linear Equations: Four Column Worksheet

Symbolic	Pictorial	What You Did	Property
			

Linear Inequalities: Four Column Worksheet

Symbolic	Pictorial	What You Did	Property

I don't have the space here to show all four columns so I will do the symbolic and pictorial only.

Notice the shift to inequalities. The same method can be used to solve equations and inequalities after doing work with context to understand the differences between them.

Try one with the algebra tiles with me, draw the two equivalent lines on your whiteboards with the inequality sign.

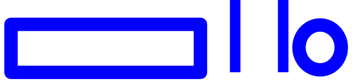
Symbolic	**Pictorial**
$2(-3x+4)+2x < -2x+2$	

Students usually do not know where to begin so I have them draw the shapes and draw the goal as the first step.

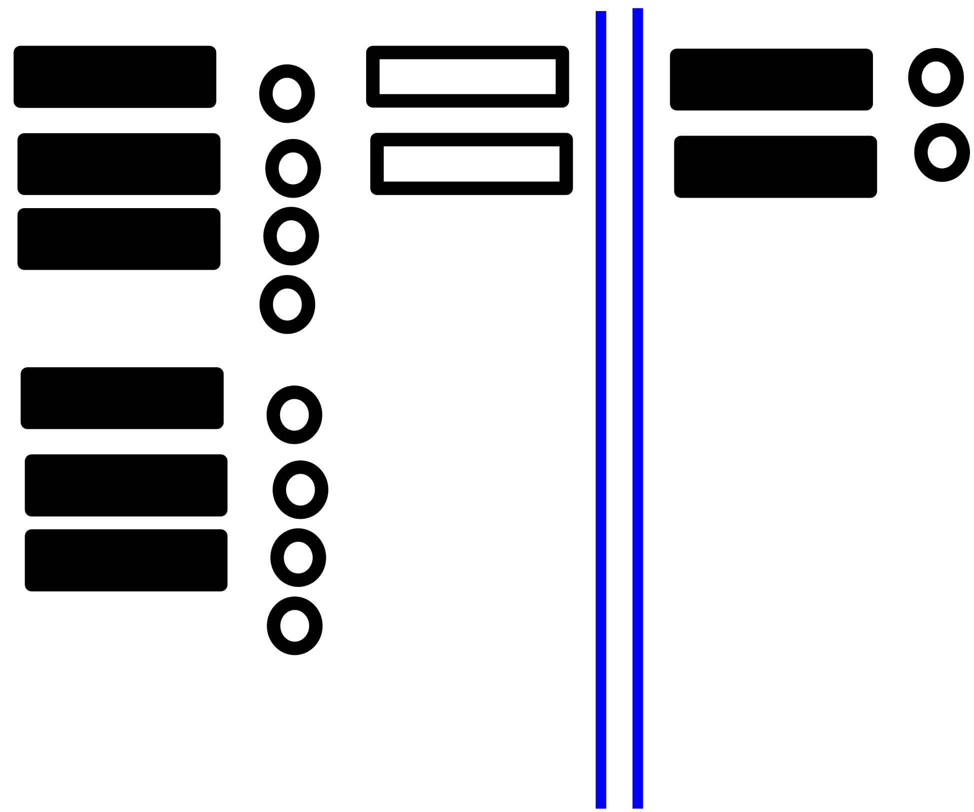
Inequalities: Given – Distributive Property

Symbolic

$2(-3x + 4) + 2x < -2x + 2$



Pictorial



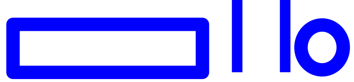
Reading the shapes horizontally is what is given.
Reading the shapes vertically is the same as distributing.

Inequalities: Given – Distributive Property

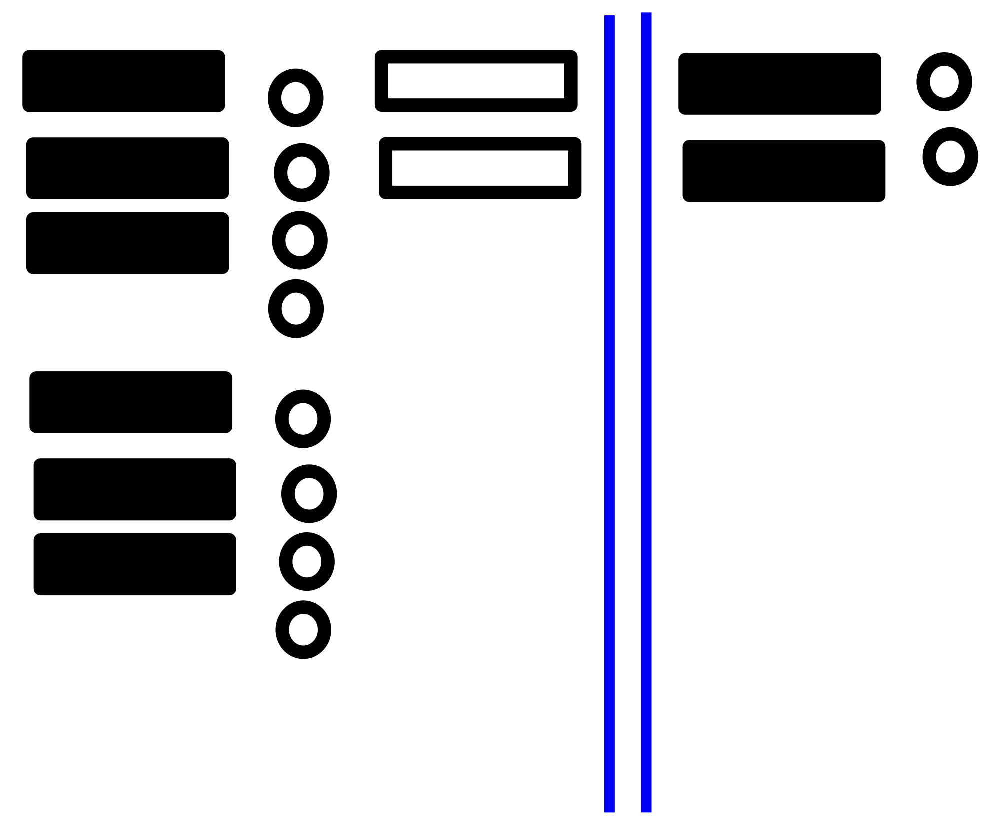
Symbolic

$2(-3x + 4) + 2x < -2x + 2$

$-6x + 8 + 2x < -2x + 2$



Pictorial



Emphasize that you have to distribute to even draw the shapes.

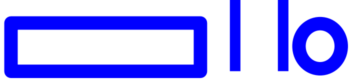
Inequalities: Combine Like Terms & Additive Inverse

Symbolic

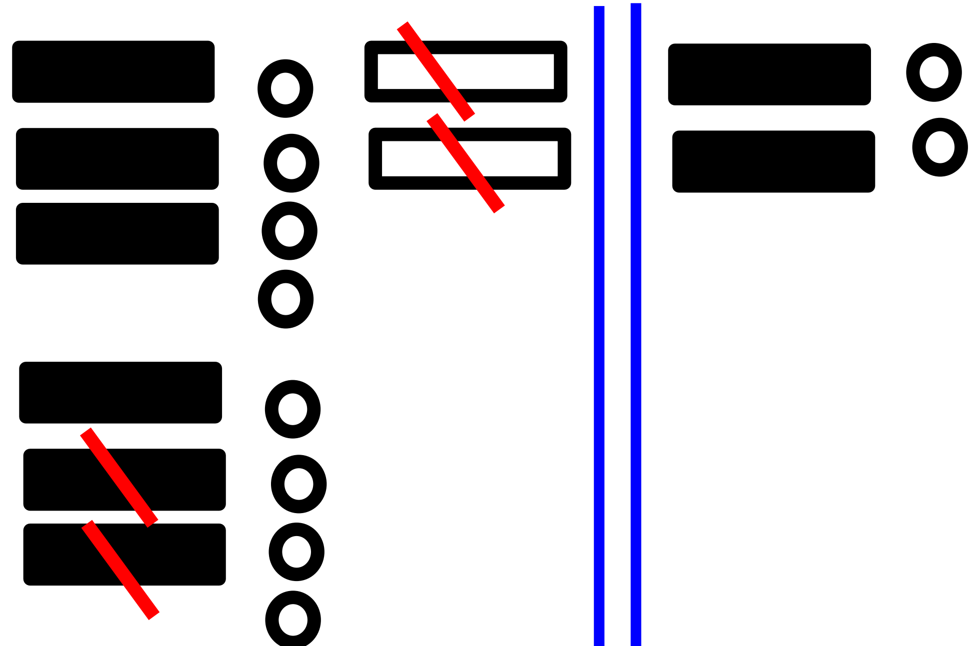
$2(-3x+4)+2x < -2x+2$

$-6x+8+2x < -2x+2$

$-4x+8 < -2x+2$

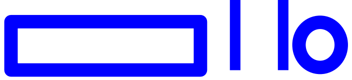


Pictorial

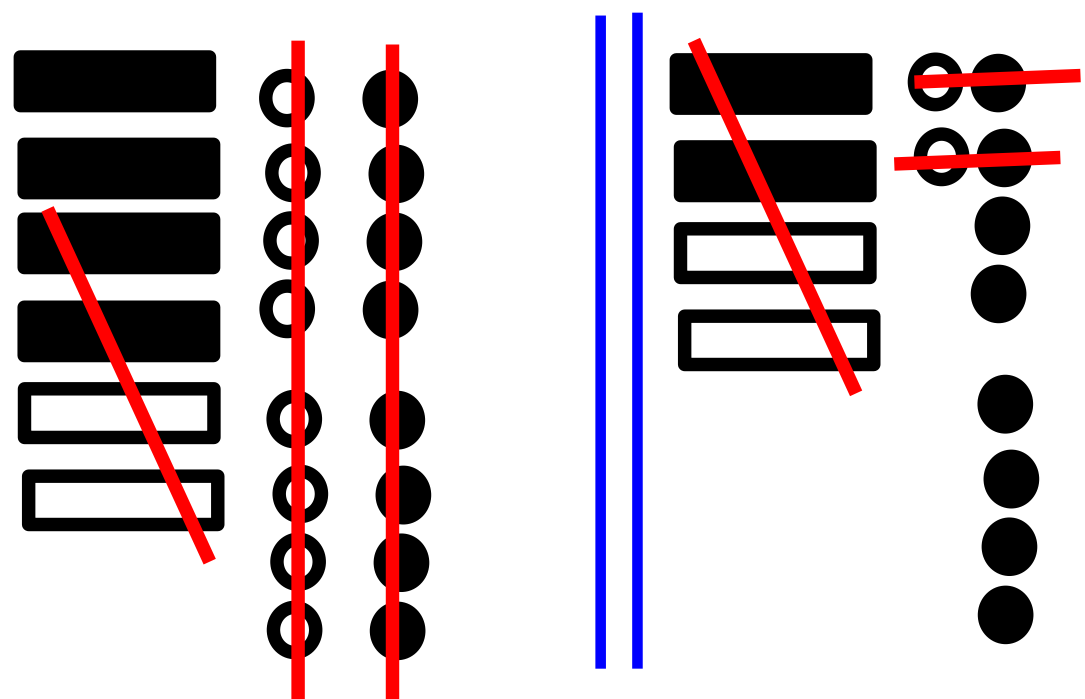


Simplify each side.

Symbolic		
$2(-3x + 4) + 2x$	$<$	$-2x + 2$
$-6x + 8 + 2x$	$<$	$-2x + 2$
$-4x + 8$	$<$	$-2x + 2$
$+2x - 8$		$+2x - 8$



Pictorial



What makes a zero pair?

Variables on one side & constants on the other.

You can do this in two steps. Once they understand the goal though, they will be comfortable with the "double switch."

Inequalities: Multiplication Property of Equality

Symbolic

$2(-3x + 4) + 2x < -2x + 2$

$-6x + 8 + 2x < -2x + 2$

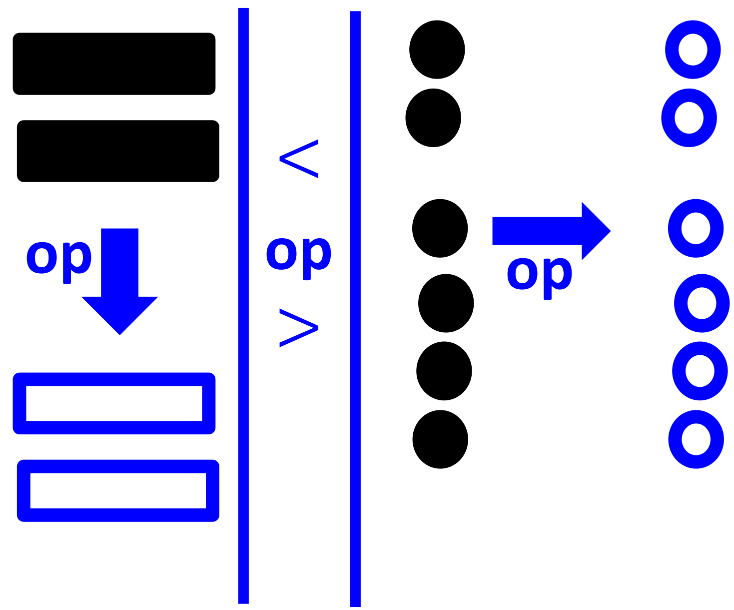
$-4x + 8 < -2x + 2$

$+2x - 8 \qquad +2x - 8$

$-2x(-1) < -6(-1)$

op op op

Pictorial

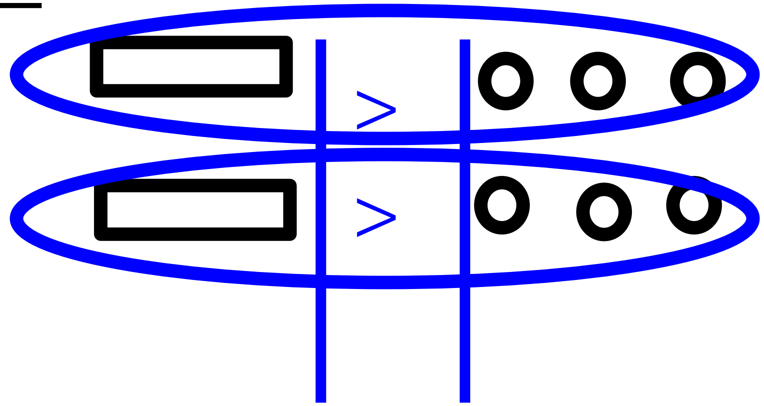


My goal is one positive unknown, x. I need the opposite of what I have so multiply both sides by -1. I have students write op-op-op, especially for knowing when to switch the signs in inequalities. Write it three times; do it three times. Notice this gets rid of any double negative division.

Inequalities: Division Property of Equality & Solution

Symbolic		
$2(-3x+4)+2x$	$<$	$-2x+2$
$-6x+8+2x$	$<$	$-2x+2$
$-4x+8$	$<$	$-2x+2$
$+2x-8$		$+2x-8$
$-2x(-1)$	$<$	$-6(-1)$
$\frac{2x}{2}$	$>$	$\frac{+6}{2}$
x	$>$	3

Pictorial



Make the coefficient one. If 2x contain 6, then how many in one?

Did I subtract? What operation did I use?

Divide by the coefficient to make a single x.

Next Level of Abstraction
Drawing Around the Symbols

$$5 + 2(-3x + 4) + 2x \geq -2x + 1$$

$$+5 \; \boxed{-6x} \; +8 \; \boxed{+2x} \geq \boxed{-2x} \; +1$$

$$\boxed{-4x} \; +13 \geq \boxed{-2x} \; +1$$

$$\boxed{+2x} \; -13 \qquad +2x \; -13$$

Same team or do they fight?

$$-2x(-1) \geq -12(-1)$$

op op op

$$\frac{2x}{2} \leq \frac{+12}{2}$$

$$\boxed{x} \leq 6$$

This is a good time to introduce larger coefficients or even rational coefficients.

Where do I begin?

You cannot draw shapes around parenthesis so distribute first to get rid of them.

Now draw your shapes & your goal.

Put variables on the left and circles on the right.

Opposite Side, Opposite Sign.

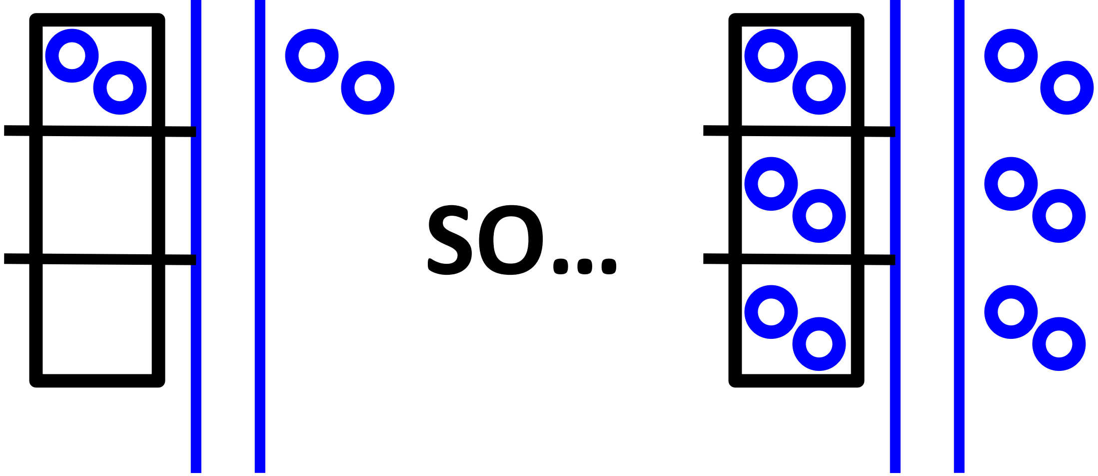
WAIT! Op-Op-Op (Multiply by Negative One)

Divide by the coefficient.

Equations: What About Unit Fractions?

$\frac{1}{3}x = 2$ $\qquad$ $(3)\frac{1}{3}x = (3)2$

$\qquad\qquad\qquad\qquad x = 6$



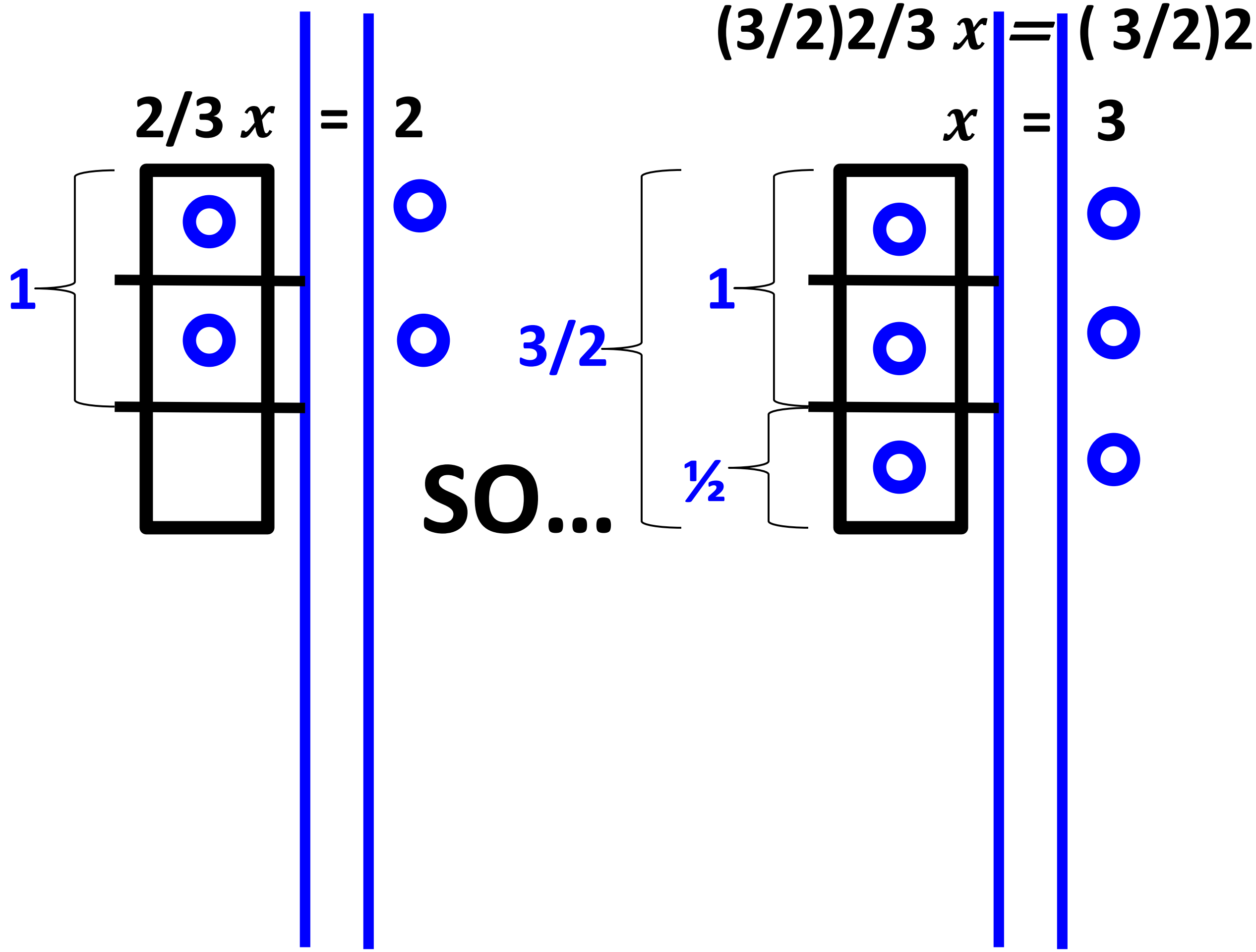
You can push for understanding by illustrating simple fractions with students.

What did I have to do to change the coefficient of one third to one?

I multiplied by 3 on both sides (Multiplication Property of Equality). What about one fourth? One fifth? Etc.

This can be extended to multiplying to get rid of fractions first.

Equations: What About Fractions?



You can push for understanding by illustrating simple fractions with students.

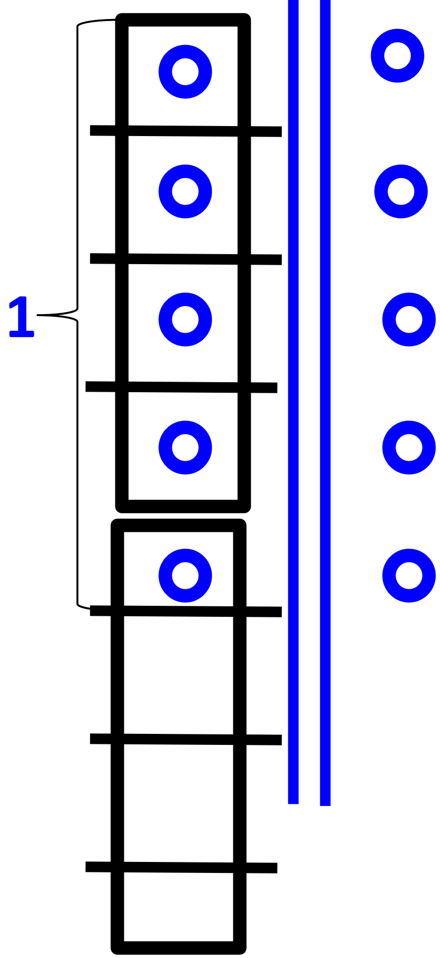
What did I have to do to change the coefficient of two thirds to one?

I multiplied by 1 ½ or 3/2 on both sides (Multiplication Property of Equality).

Does the reciprocal always work? What about improper fractions?

Equations:
What About Improper Fractions?

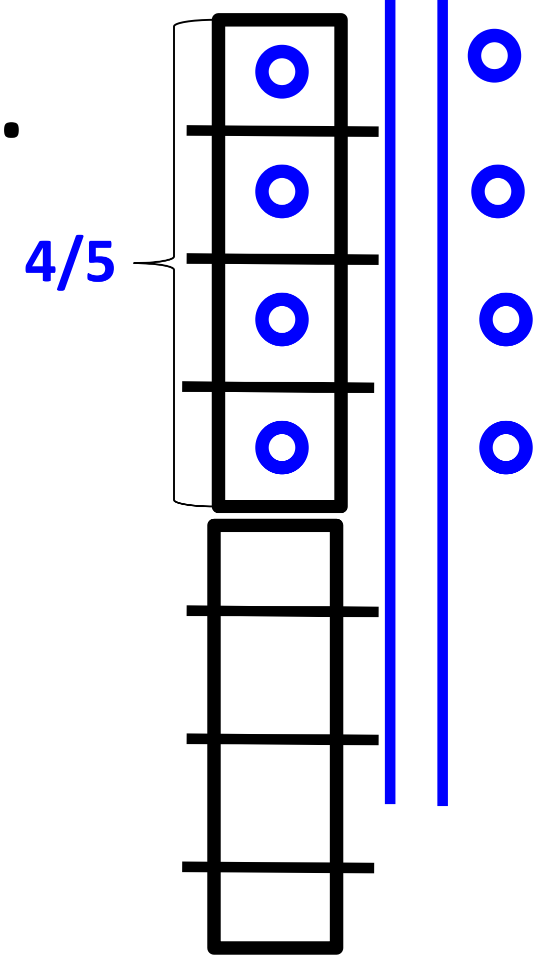
$5/4\,x = 5$



SO…

$(4/5)5/4\,x = (4/5)5$

$x = 4$



What did I have to do to change the coefficient of five fourths to one?

I multiplied by 4/5 on both sides (Multiplication Property of Equality).

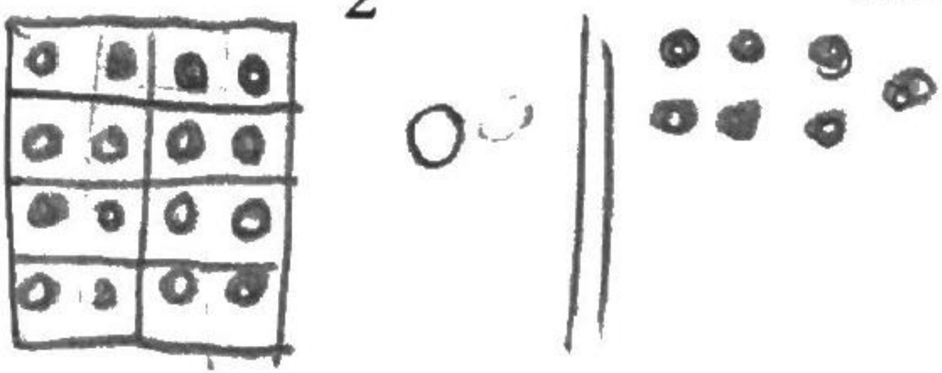
Would this be a good way to build some understanding of fractions without interrupting your teaching?

Equations:
Student Work

What does this student understand about equality?

Division? Integers? Fractions?

8. $-7 = \frac{z}{2} + 1$



$\boxed{z = -16}$

$\frac{-16}{2} = -8$

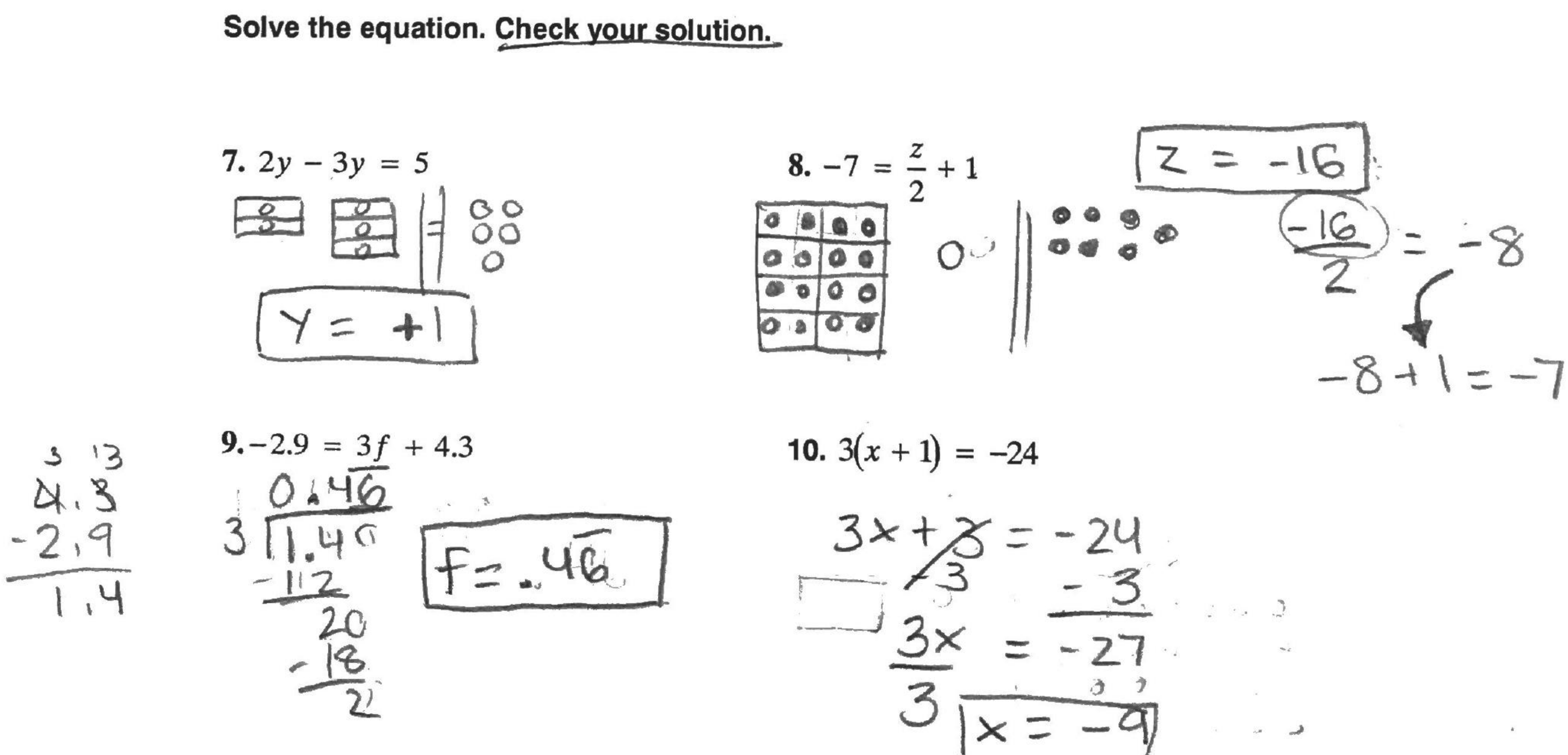
$-8 + 1 = -7$

Equations: Student Work
Was she always doing that?
What about necessary procedures?

She knows and uses procedures too.

We don't know about fractions… not ready for the next level of abstraction?

Does she find this easier/faster?



Solve the equation. Check your solution.

7. $2y - 3y = 5$

$\boxed{Y = +1}$

8. $-7 = \frac{z}{2} + 1$

$\boxed{Z = -16}$

$\frac{-16}{2} = -8$

$-8 + 1 = -7$

9. $-2.9 = 3f + 4.3$

$4.3 - 2.9 = 1.4$

$1.4 \div 3 = 0.4\overline{6}$

$\boxed{f = .4\overline{6}}$

10. $3(x + 1) = -24$

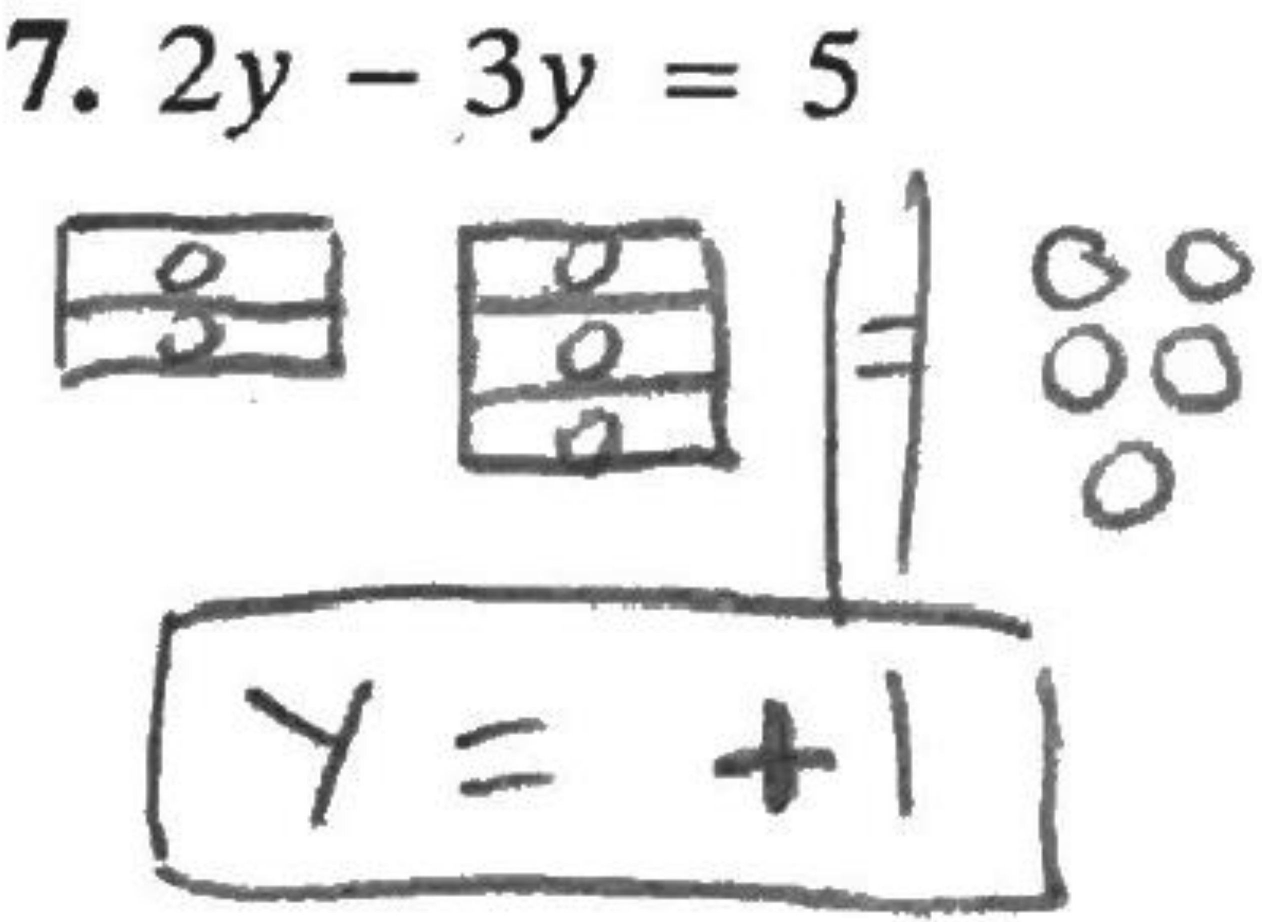
$3x + \cancel{3} = -24$

$\cancel{-3} \quad -3$

$\frac{3x}{3} = -27$

$\boxed{x = -9}$

Equations: Student Work
It is not all roses...



What is the mistake?

The student disregarded the – sign and/or failed to combine like terms.

What reengagement lesson would you design around this problem?

How could you use this student's method of substitution or guess and check correctly? (Hint: Negative means opposite.)

Giving Students the Procedures?

Students should <u>generalize to procedures themselves</u> so they are more likely to remember them.

If you must provide them, then they should be a <u>temporary</u> resource for students. Keep them <u>simple</u>, <u>short</u> and <u>always true</u>:

1. **Draw the shapes and the goal [this will help them begin]**
2. **If there are parenthesis, then you don't have a single group so you cannot draw the shapes. You must distribute first, if necessary. 3(2x +1) is not equivalent to 32x + 1, NOT $\boxed{3(2x}$ +1)**
   While this next one is technically correct, it loses the meaning of the three groups, NOT 3 ($\boxed{2x}$ +1)
3. **Simplify each equivalent expression, if possible.**
4. **Collect variables on one side & constants on the other.**
5. **Make the coefficient of the variable one.**

What Materials Did I Use?

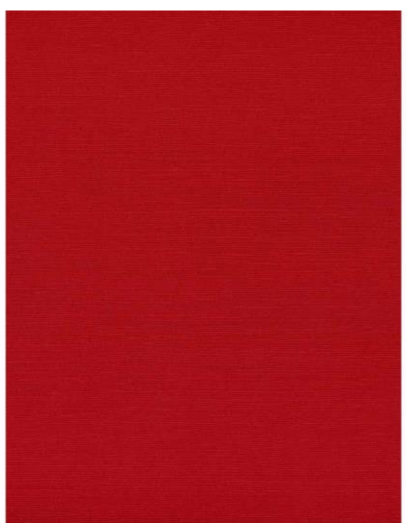
Two colors of cardstock

Slider bags

Folded blank paper

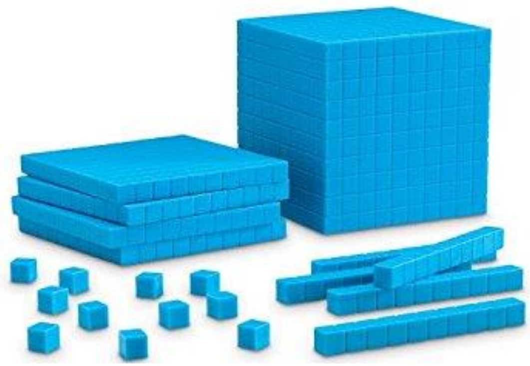
Cheapest manipulatives EVER!

And they do much more than this…



What about the Base Ten Blocks?

Some might argue against having the same manipulative as base ten and algebra tiles.



My counter argument is why keep two manipulatives when one will do?

More importantly, I *want* them to make connections between base ten and algebraic representations of any base to aid in understanding and retention.

Color and Algebra Tiles?

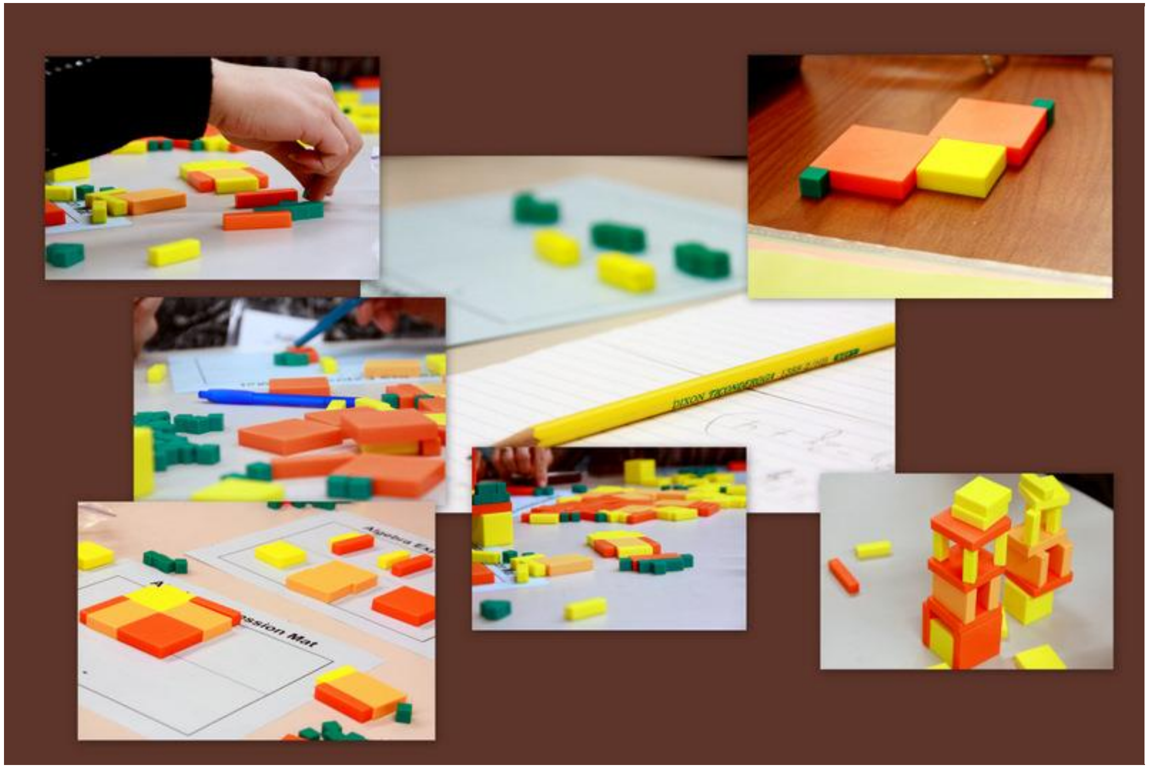
Store bought algebra tiles have “unknown” side lengths as they are trying to reinforce the that quality of the variable.

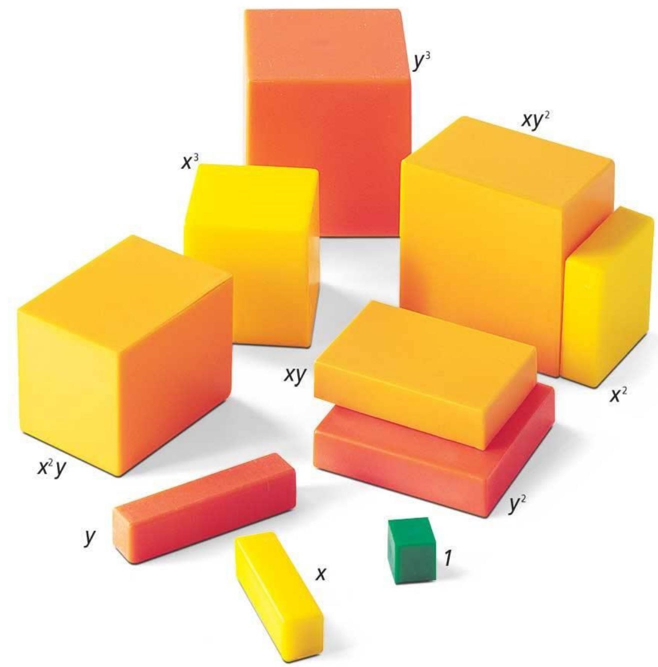
If you measure these tiles with the one, then you find they are usually between three and four. Does that mean it has a fractional side length?

All models have inherent flaws; choose the flaws that reinforce the learning when you can.

I don’t prefer the multiple color tiles. I prefer to use color to distinguish between positive and negative or different variables.

What about Y? Color & Algeblocks





AlgeBlocks allow for work in three dimensions, cubic expressions & equations, and work with two variables, x & y.

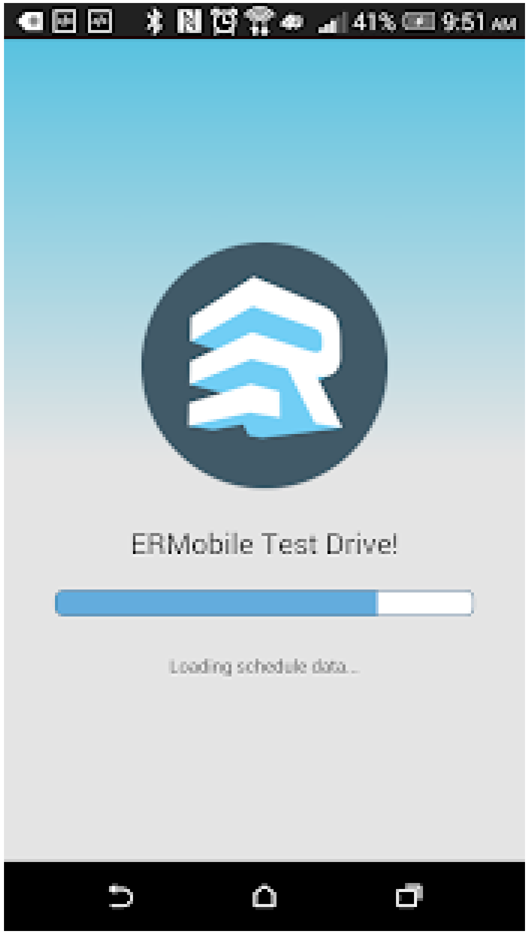
http://www.hand2mind.com/item/algeblocks-manipulative-starter-set/9241 Expensive @ $200 though.

Presentation, Questions & Feedback

Please complete the session survey via the ERMobile Conference App.

This presentation along with others are posted on my fledgling website www.meaningformemory.com

Please leave other feedback & questions on the site or email me at david.mattoon@gmail.com



Handouts

David Mattoon

Meaning for Memory

What are our words worth?

"I'll give you a thousand words for it."

A Picture is Worth a Thousand Words ~ Fred Barnard, 1927

The notion that a **complex idea** can be conveyed with just **a single still image** or that **an image of a subject conveys its meaning** or essence **more effectively** than a description does.

Ever get tired of repeating yourself ?

- **"Hearing something a hundred times isn't better than seeing it once"**

  ~ Chinese Expression
- **“The drawing shows me at a glance what would be spread over ten pages in a book.”** ~ Ivan Turgenev, Russian Author
- **"A good sketch is better than a long speech“** ~ Napoleon Bonaparte, Emperor of the French