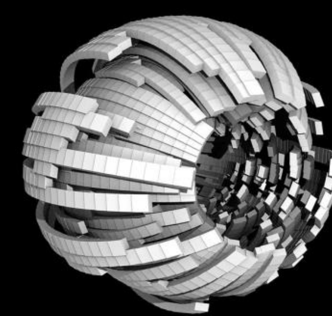
What are our words worth?

"I'll give you a thousand words for it."

Algebra as Area: Polynomial Multiplication & Factoring

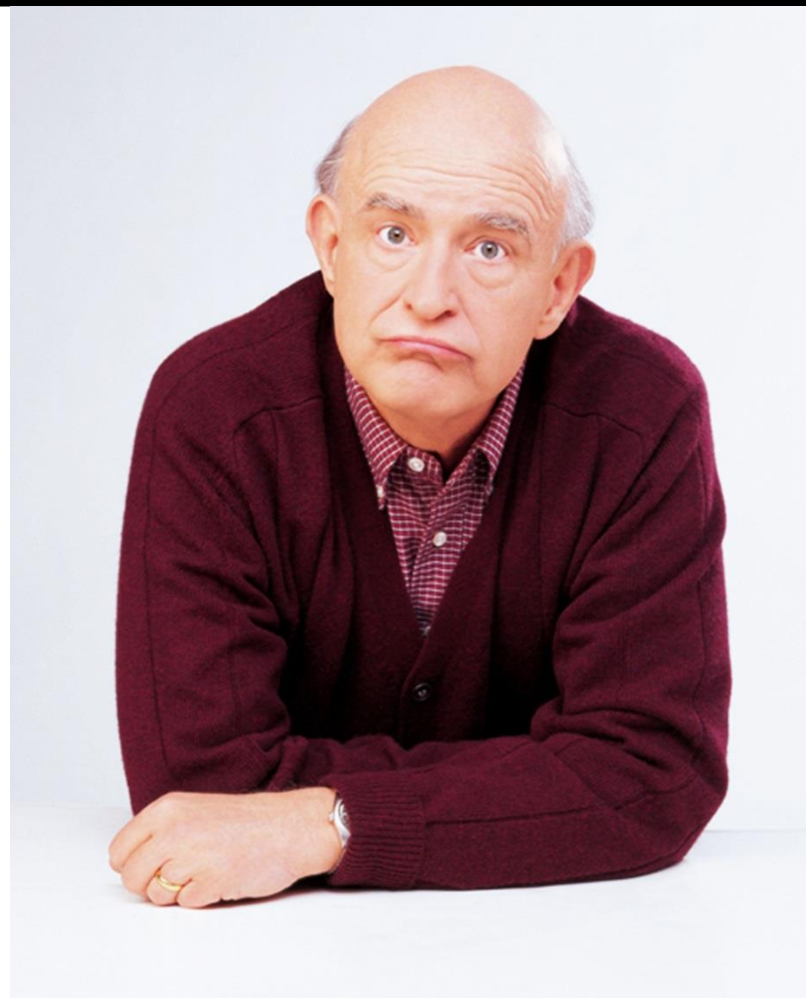


David Mattoon

Meaning for Memory

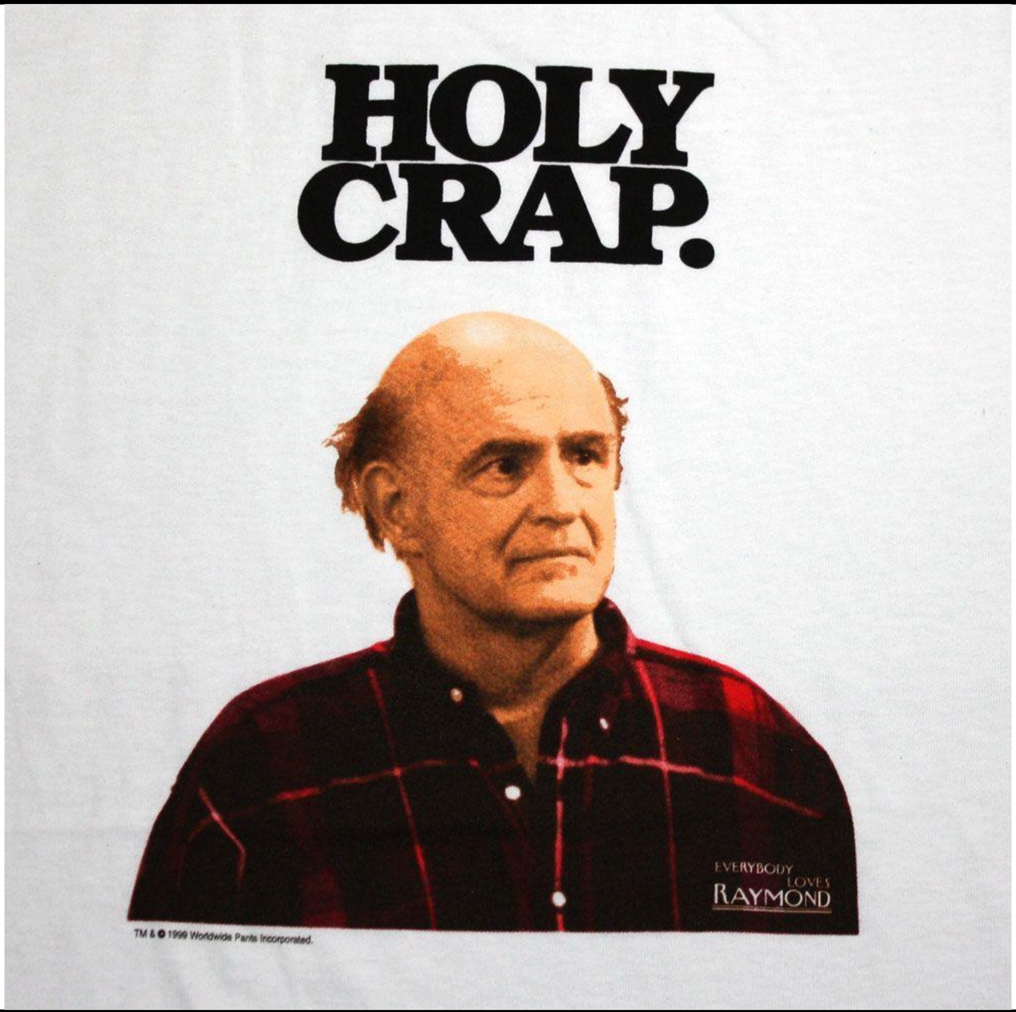
2018 CMC-South Annual Conference

My Frank Barone Moment

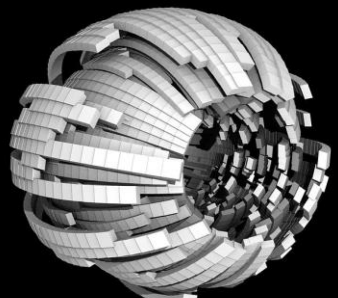


David Mattoon Meaning for Memory

My Frank Barone Moment



Deficiency
Time
Concreteness Fading
Rigor



David Mattoon
Meaning for Memory

Deficiency: Students or Environment?

In educational settings, this construction of dis/ability manifests in the double education system that splits general education and special education. Scholars have traced the ways in which special education "**serves as a vehicle for preserving general education in the midst of ever increasing diversity**" (Reid & Valle, 2004, p. 468, paraphrasing Dudley-Marling, 2001; also see Skrtic, 1991, 2005). **Rather than using research-validated frameworks** like Universal Design for Learning (UDL) and Complex Instruction (CI) **to deliver rigorous, high-cognitive demand instruction to all mathematics students**, the system of special education **shunts certain students (especially students of color) into an inferior, segregated mathematics education, thus providing a band-aid to a broken general education system and preventing larger, more systematic changes.**

Returning to the assumptions inherent in the concept of intervention, a disability studies perspective problematizes **the taken-for-granted assumption that what is "wrong" with the situation requiring intervention is a pathology or deficit within students. Instead, the problem is located in the inaccessibility of the environment; in other words, what needs to be changed is not the student, but rather the environment to allow access for students who differ from one another.** As Reid and Valle (2004) assert, "the responsibility for 'fitting in' has more to do with changing public attitudes and the development of welcoming classroom communities and with compensatory and differentiated instructional approaches than with individual learners (Shapiro, 1999). In other words, **our focus is on redesigning the context, not on 'curing' or 'remediating' individuals' impairments"** (p. 468). A related line of research of working group members involves conceptualizing interventions into participation rather than content. That is, **what interventions might contribute to more equitable participation and deeper engagement across students in mathematics classrooms?** For example, one of the working group members has conducted empirical research focused on equitable participation in a Cognitively Guided Instruction algebra routine. Moreover, a political/relational model suggests that inaccessibility is embedded in the context of power relations. **Finding ways to "intervene" to make the environment accessible, then, also requires analyzing the power relations involved** in maintaining inaccessibility.

http://jamessheldon.com/reframing-interventions-in-mathematics-education-emerging-critical-perspectives/

Change the Environment

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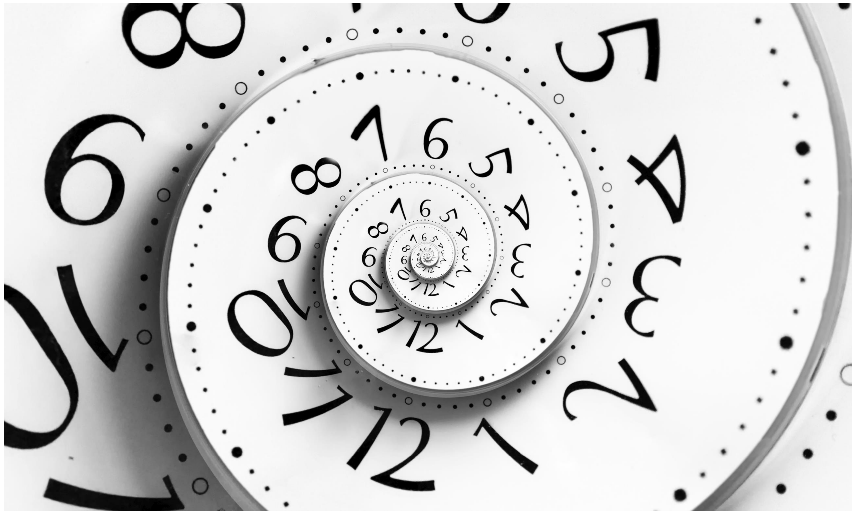
Time: I don't have it.

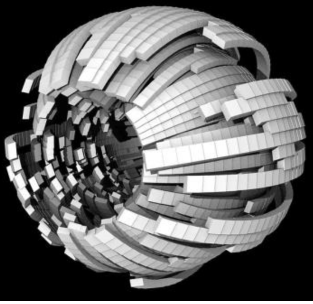
I might make time to…



Pictorial Math will save you time in the long run as it will aid students in understanding what to do, why they do it, and how to transfer the knowledge.

- Review in the beginning of the year
- Introduce new material by re-teaching material from former grade levels
  - I am not talking about making connections to former material here.
- Reteach for full periods when they don't understand
- Reteach for full periods when they cannot remember
  - I am not talking about interleaving
- Teach multiple procedures for slightly different problems
  - I am not saying procedures are bad.

Levels of Abstraction = Concreteness Fading
Concrete, Representational, Abstract (CRA)



During the first half of my teaching career, I would spend what seemed to be the first half of a math lesson teaching a new math concept by sharing definitions, formulas, steps and procedures.

To make things more challenging for my students, I would simultaneously introduce the symbolic notation used to represent those ideas. Then, I would spend the **remainder** of the lesson **attempting** to help my students make sense of these very **new** and often **abstract ideas**.

By the end of the lesson, I could help many students build an understanding, but there was **always a group I felt who I would leave behind.** Like many other teachers, I was just **teaching in a very similar way to that how I was taught**. **I knew no different.**

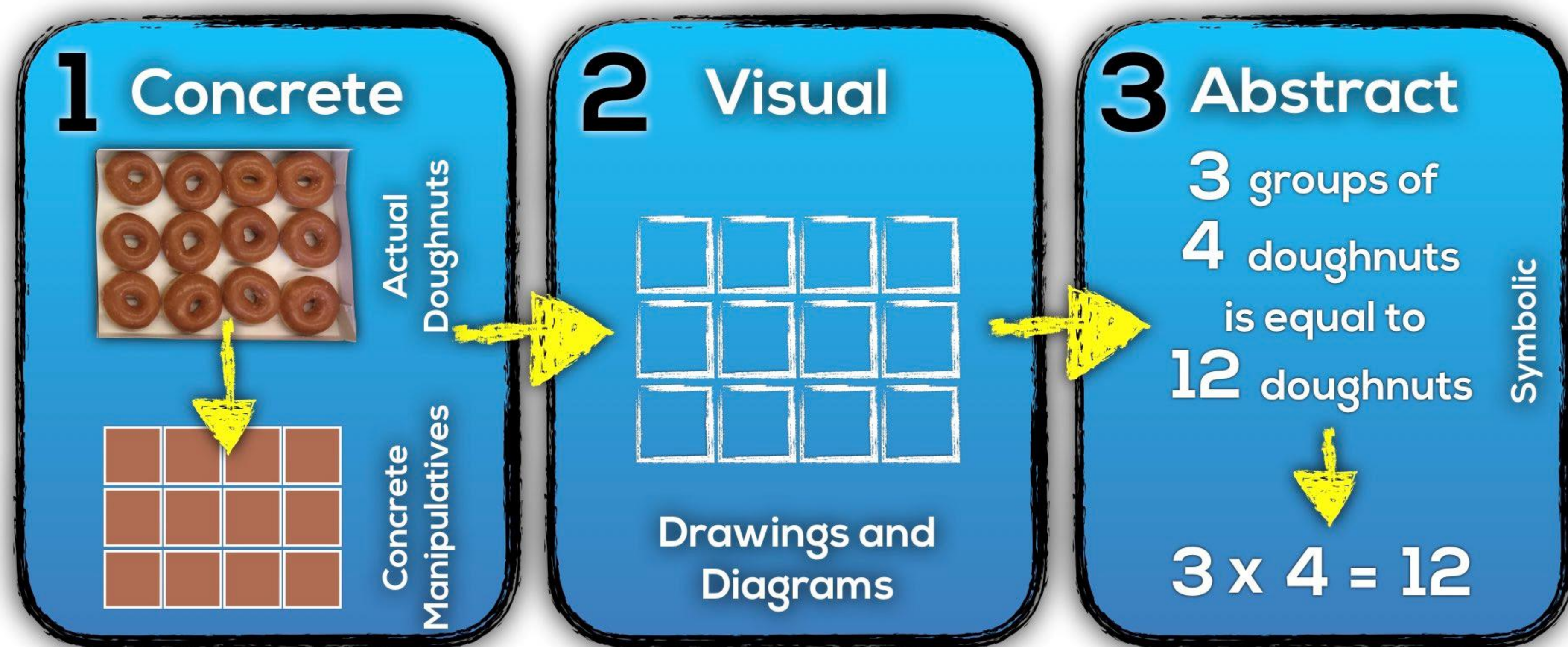
However, if we consider that new learning requires the linking of new information with information they already know and understand, we should be intentionally planning our lessons with this in mind. **A great place to start new learning is through the use of a meaningful context and utilizing concrete manipulatives that students can touch and feel.**

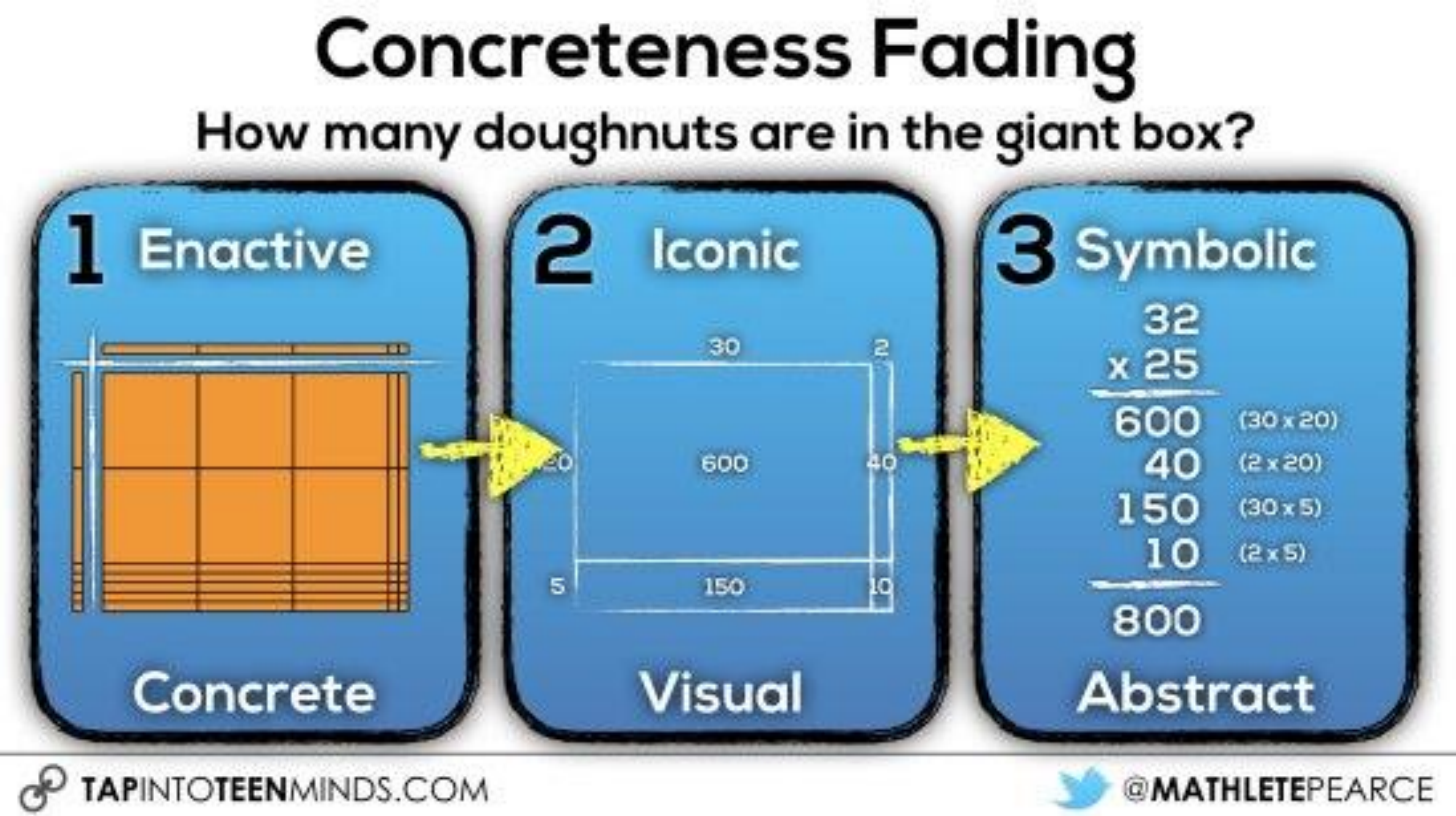
When we teach in this way, we **minimize the level of abstraction** so students can **focus their working memory on the new idea** being introduced in a **meaningful way.**

- Kyle Pierce, Tap into Teen Minds, https://tapintoteenminds.com/concreteness-fading/

Concreteness Fading

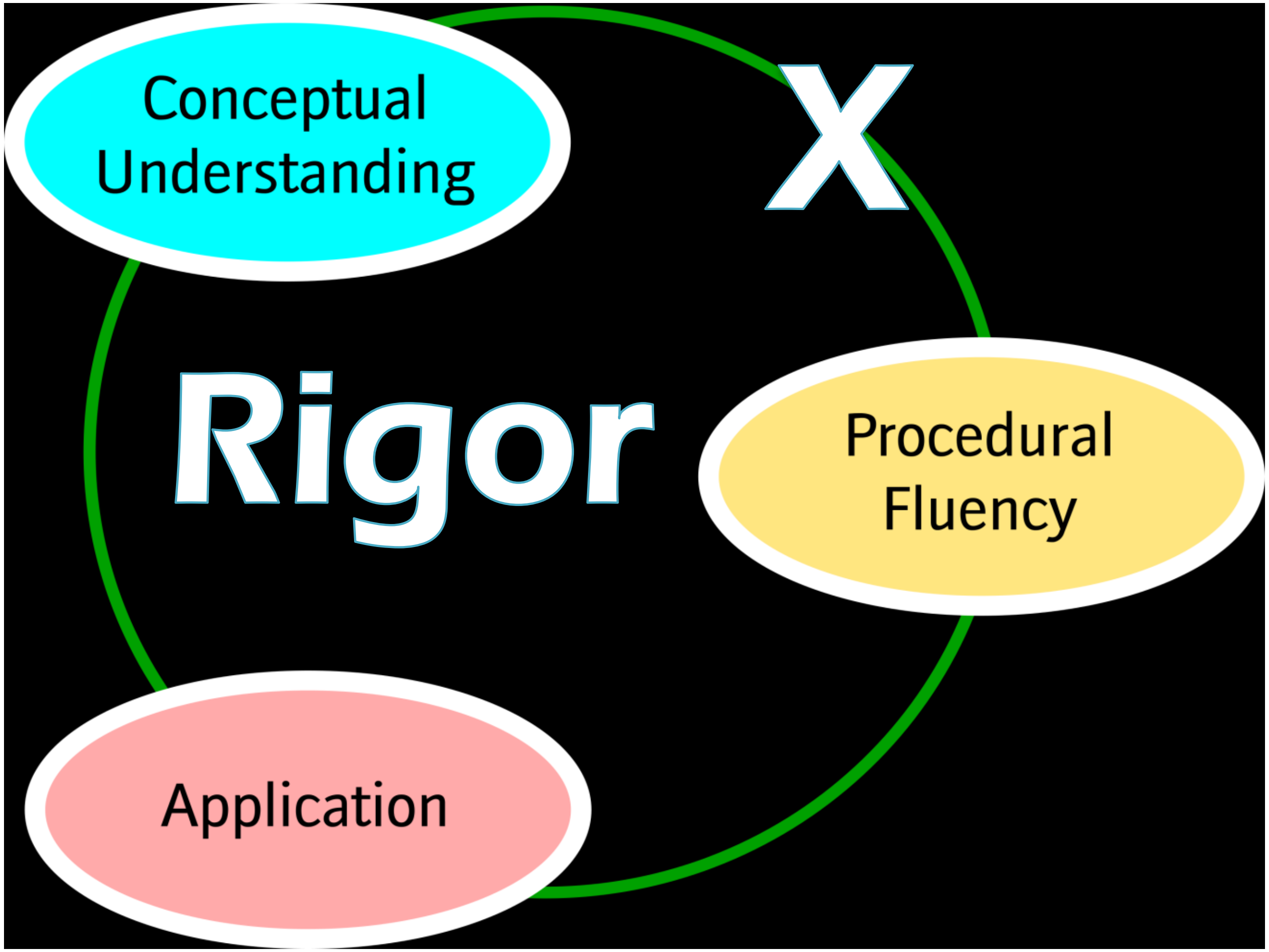
How many donuts are in 4 boxes of 12 donuts?





Jerome Bruner (1966) proposed three modes of representation:
Enactive representation (action-based)
Iconic representation (image-based)
Symbolic representation (language-based)

Where does this presentation live?

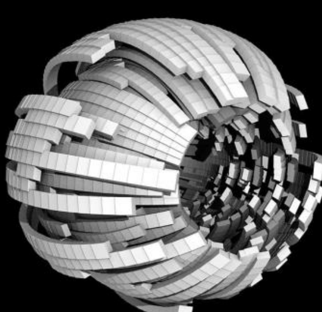


Agenda

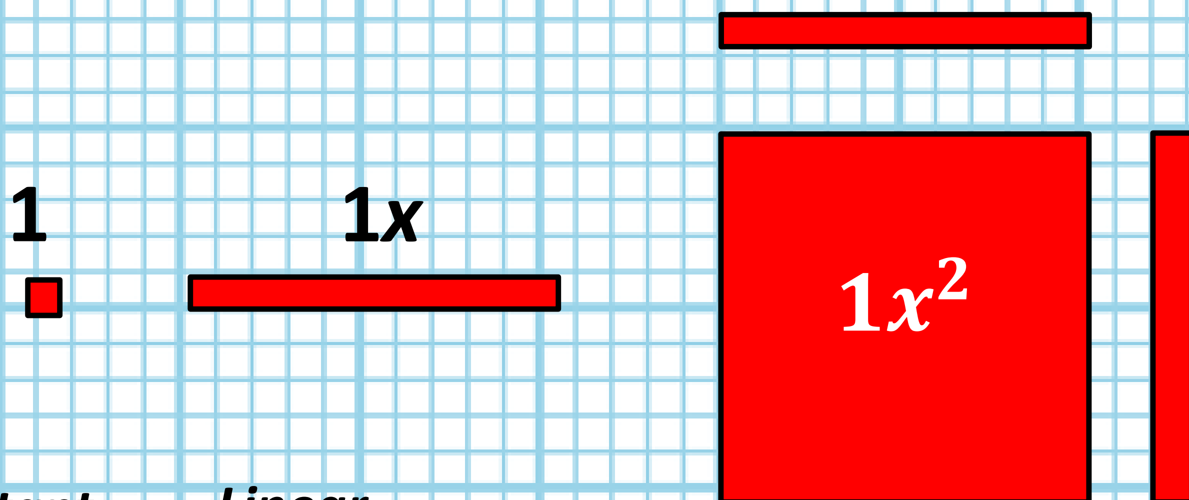
- ✓ Deficiency, Time, Concreteness Fading & Rigor
- ❑ Polynomial Combination: A Necessary First Step
- ❑ Polynomial Multiplication
- ❑ Polynomial Factoring
- ❑ Special Cases Preview

Polynomial Combination: A Necessary First Step

David Mattoon
Meaning for Memory



Algebra Tiles



Constant –

a length of 1 and a height of 1

Linear –

a line of length x and a height of 1

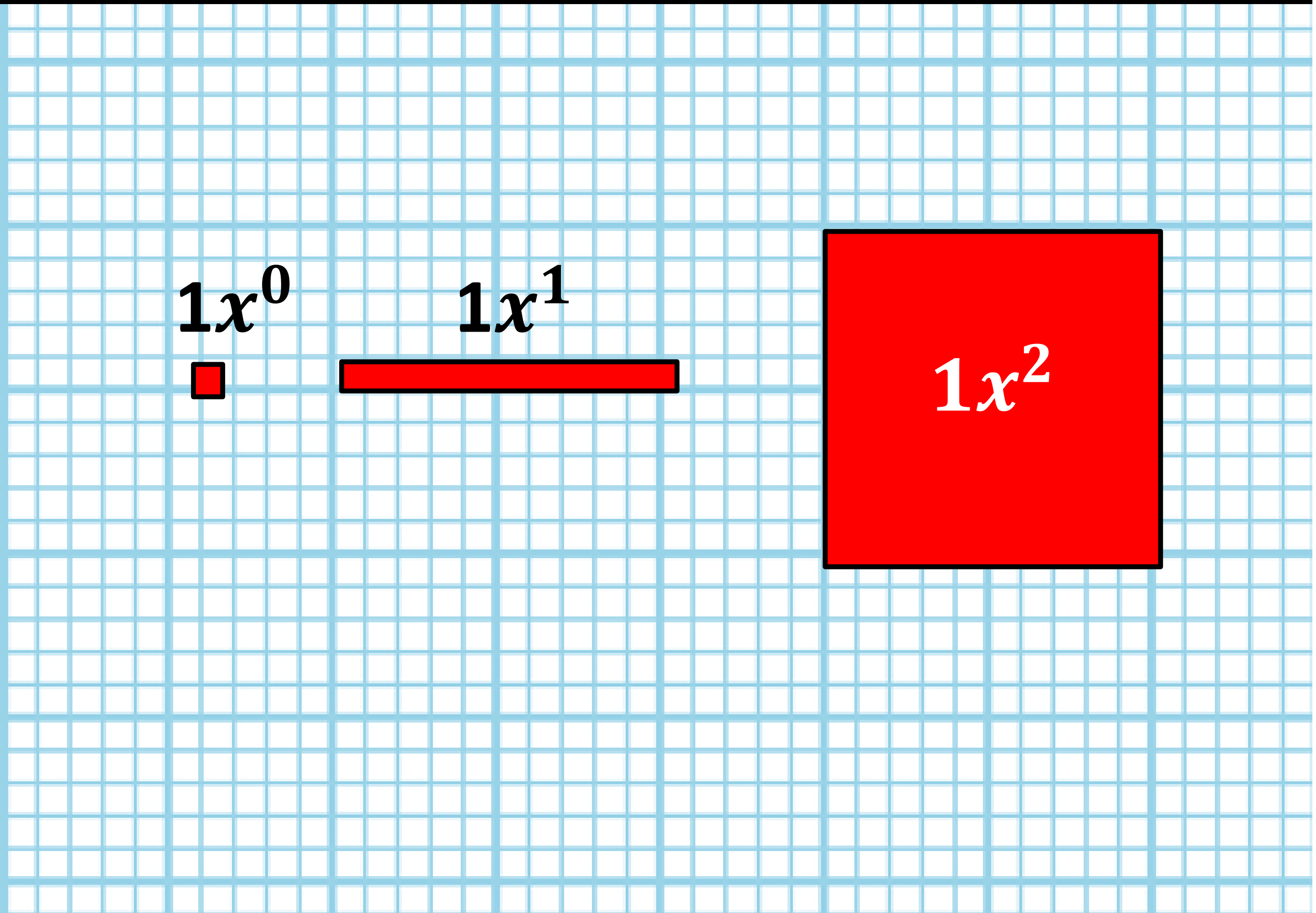
Quadratic–

a square with a side length of x.

In the case of algebra tiles, one is the small square, which is the same as base ten. You might say it is constantly one. Really it is a constant , because the value of x will not influence it.

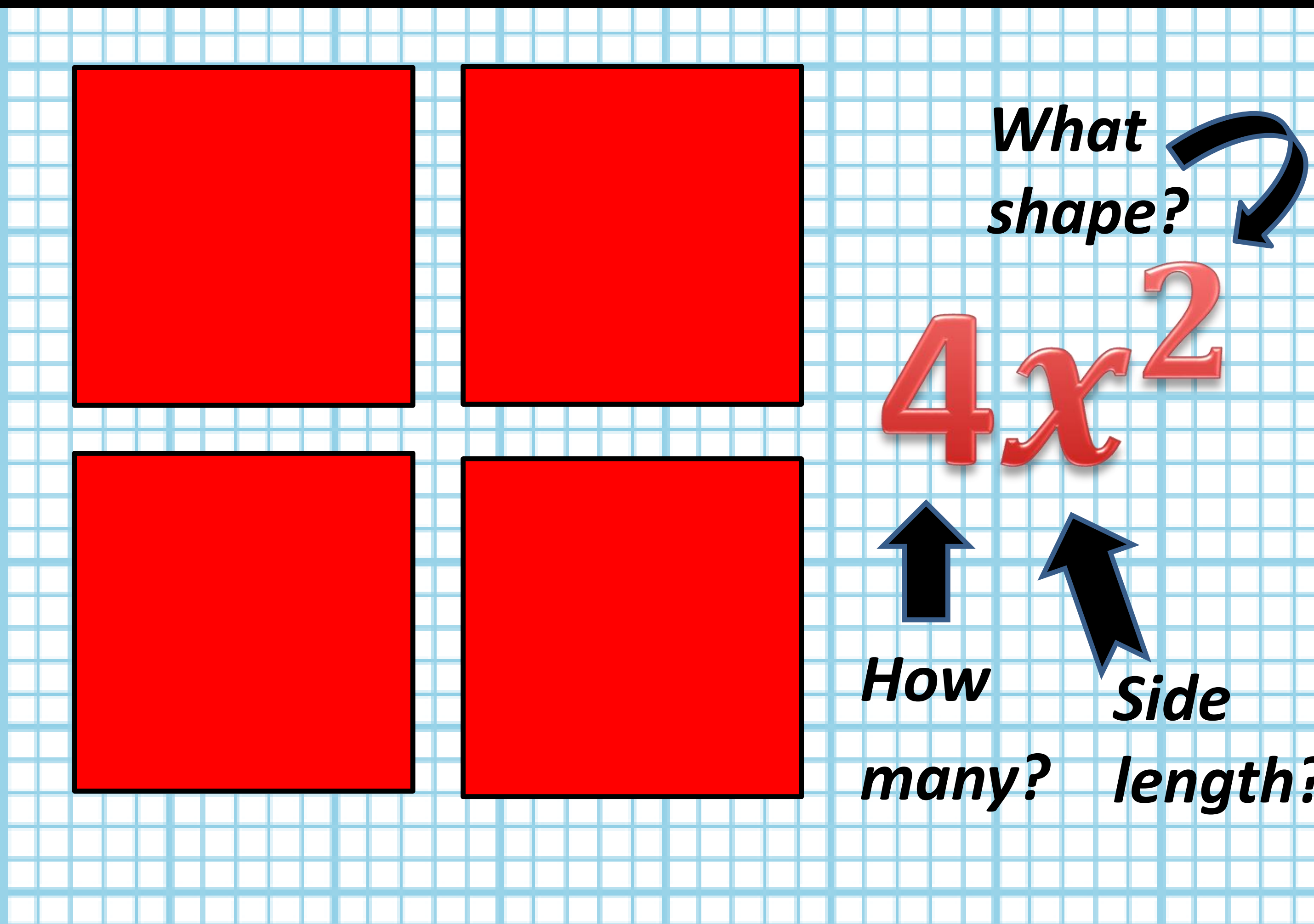
Begin referring to x squared as a square with a side length of x. A square with two dimensions, a length of x and a width of x.

Algebra Tiles



Our number system is based on powers of ten; therefore, it is the exponent, which drives what shape it is.

Anatomy of an Algebraic Term



You want students to discover this and/or be able to tell you when you ask:

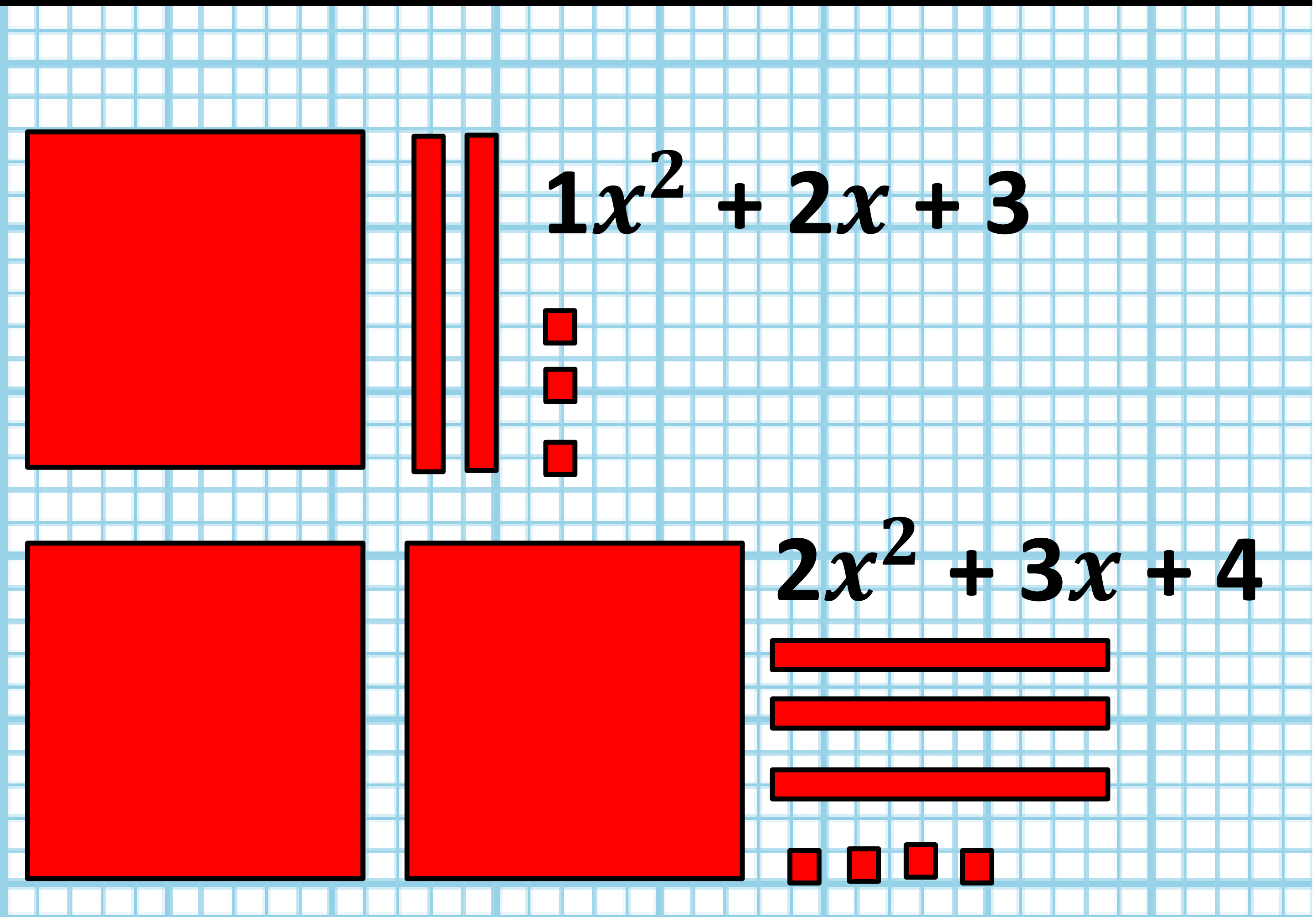
What does the exponent mean?

What does the base mean?

What does the coefficient mean?

Algebra Tiles: Addition

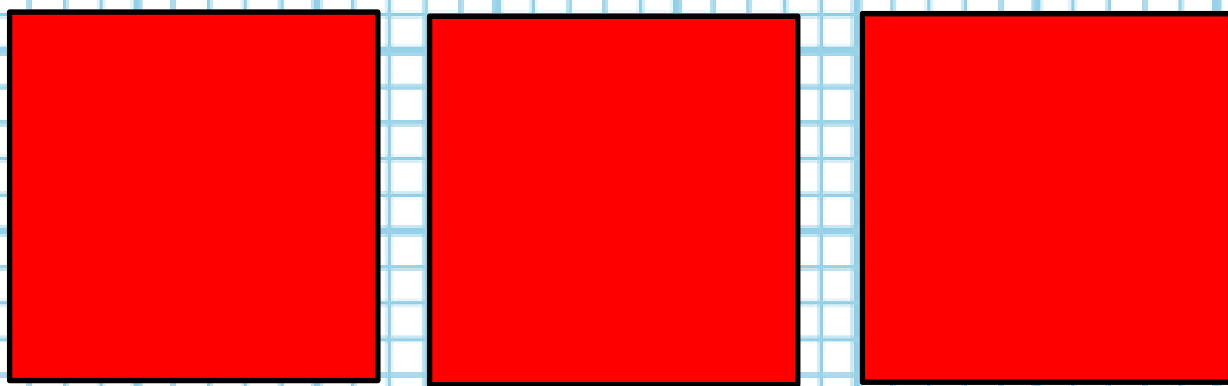
$$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$$

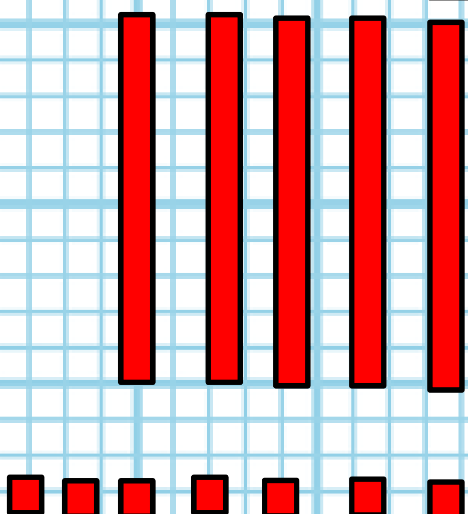


You should do some combining of polynomials with students before beginning multiplication with algebra tiles as they will need to understand what the tiles mean and how to combine like terms, terms of the same shape or power.

Algebra Tiles: Addition

$$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$$





$$\begin{array}{r} 1(x^2) + 2(x) + 3(1) \\ + \; 2(x^2) + 3(x) + 4(1) \\ \hline 3(x^2) + 5(x) + 7(1) \end{array}$$

$$\boxed{3x^2}\boxed{+ 5x} + 7$$

Have them put the same shapes together, aka combine like terms.

Can you put the terms together? No, they are not the same shape. Squares, Lines & Dots or Quadratic Terms, Linear Terms & Constants

Cognitive Demand & Access

"My students aren't ready for algebra."

"They have to master basic skills first before learning algebra."

While it is preferred to see algebra as a generalization of any base, could we use algebra to remediate number? Instead of going specific to general could we go general to specific?

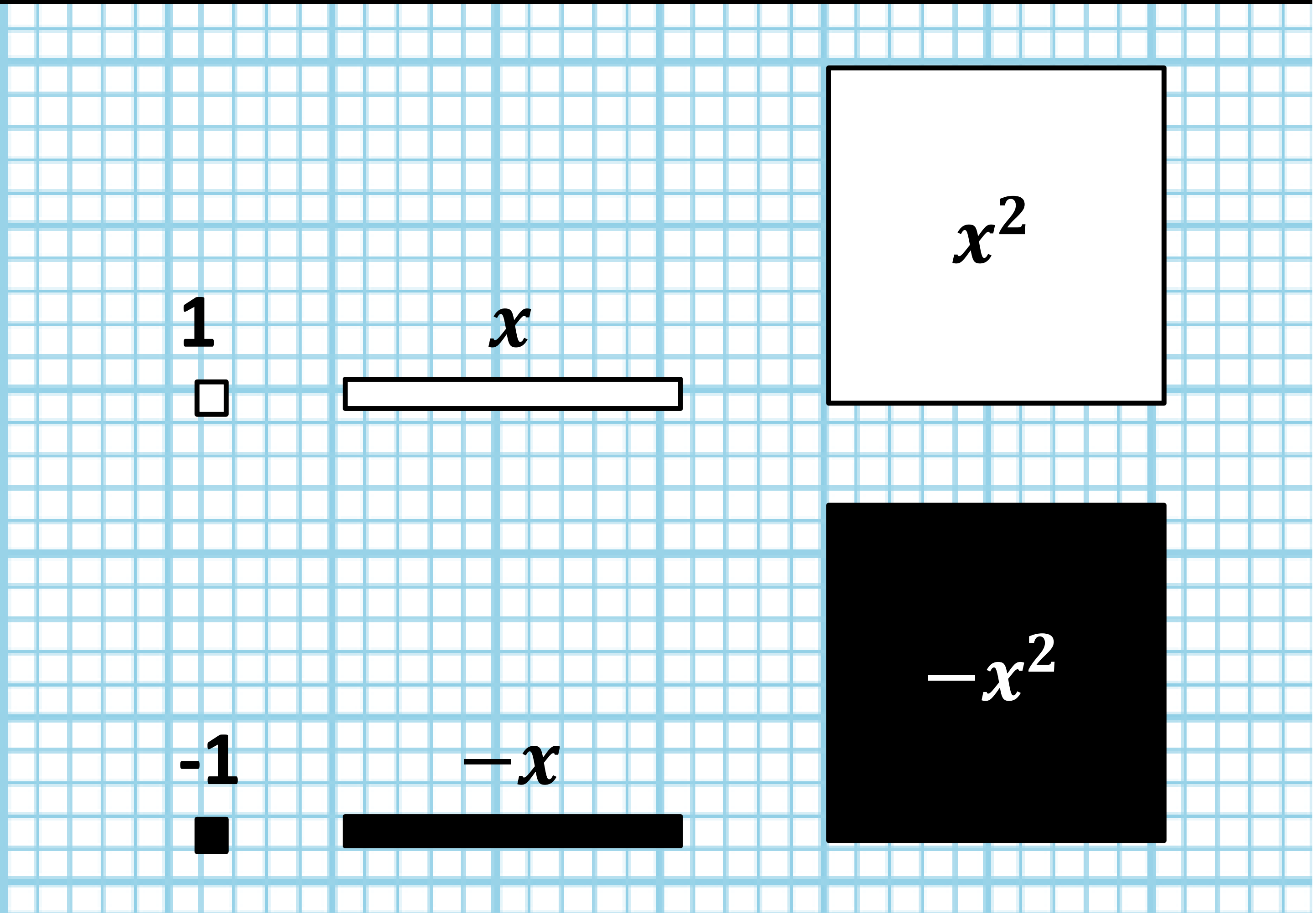
Keep students on grade level and remediate number with Tier Two interventions using more time to relate number to algebra. In class, use polynomial addition & subtraction to remediate integers or even whole numbers.

Remediate in context of grade level material.

There is no regrouping with algebra tiles as you do not know what the base is, or how many units it takes to make the units of the next power.

The lack of regrouping actually makes the cognitive demand for learning algebra less than the cognitive demand for number when using algebra tiles and base ten blocks.

Next Level of Abstraction: Drawing the Tiles Under the Expressions



"Open" is positive.

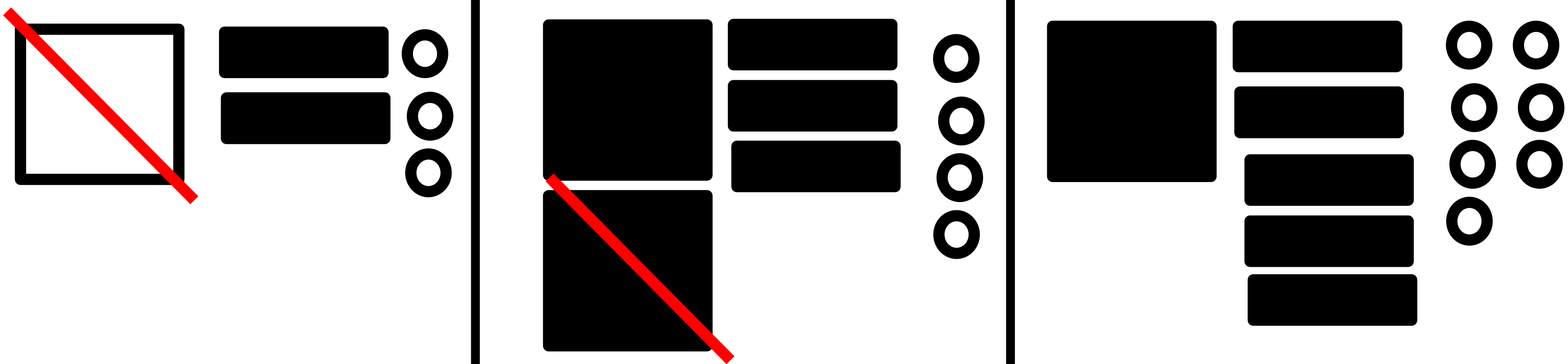
"Closed" is negative.

Referring to them as white being positive and black being negative is not a good idea.

For a full discussion of polynomial concreteness fading including subtraction. See the slides from the morning session posted at [meaning for memory.com]

Combining Expressions: Three Column Notes

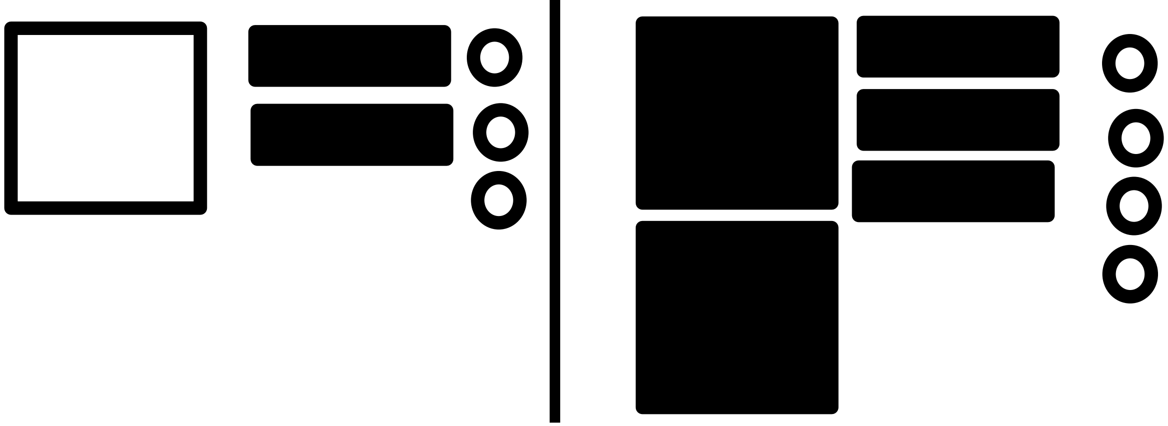
$$1x^2 - 2x + 3 \quad + \quad -2x^2 - 3x + 4 \quad = \quad -1x^2 - 5x + 7$$



Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

	+	=
$1x^2$ -2x + 3	-2x^2 -3x + 4	

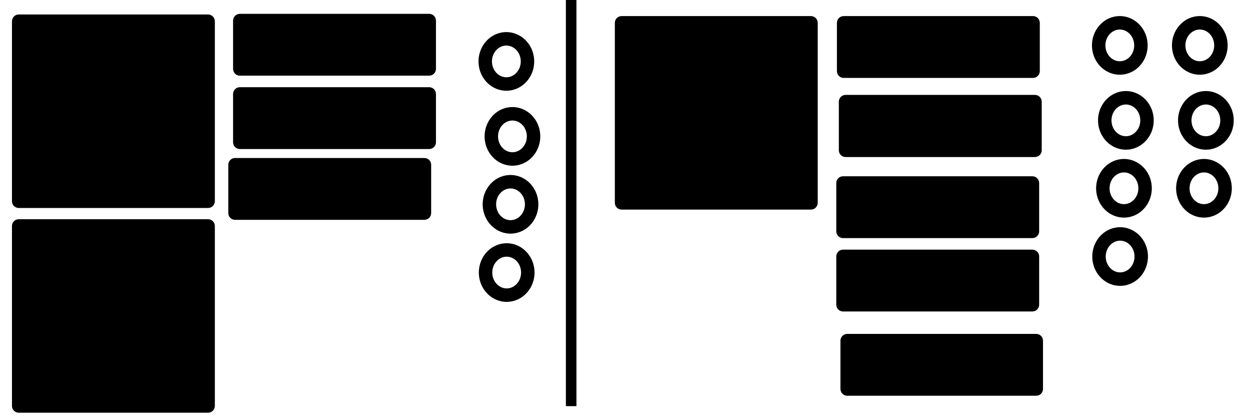


Eliminate just enough information to make it doable.

Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

$1x^2$ -2x + 3	+	= -1x^2 -5x +7



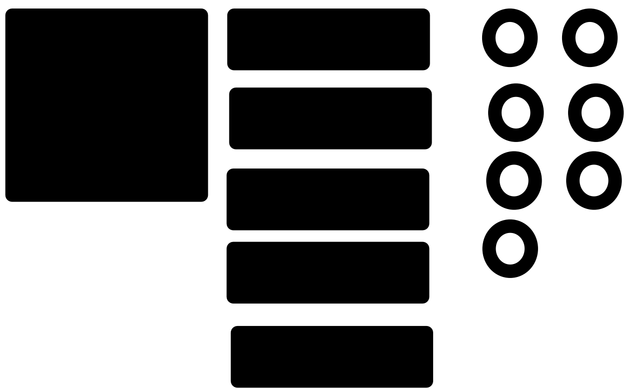
Eliminate just enough information to make it doable.

Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

$+$ | $=$

$-1x^2 - 5x + 7$

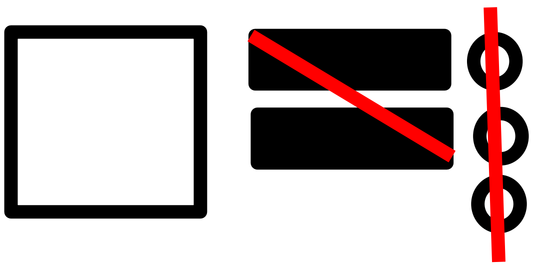


Students make their own problem.

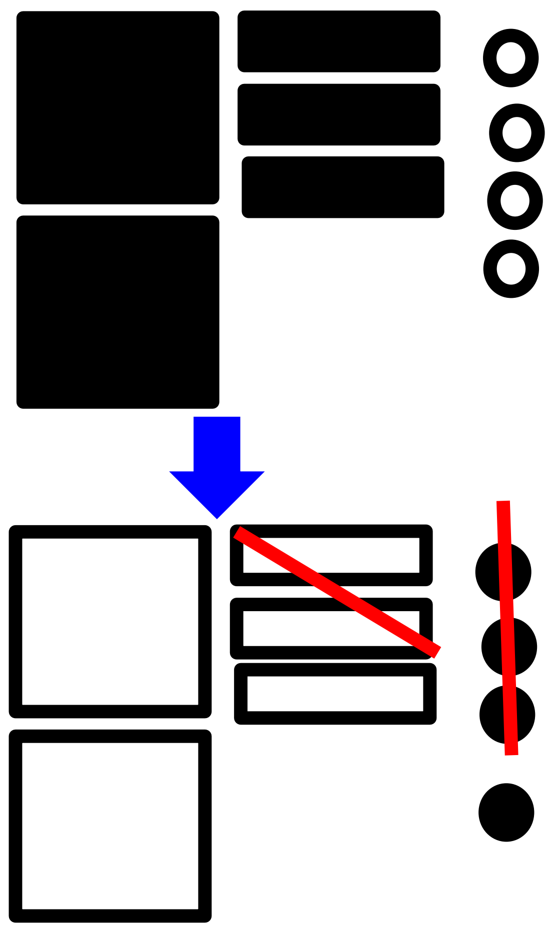
Require students to draw the shapes AND write the symbols.

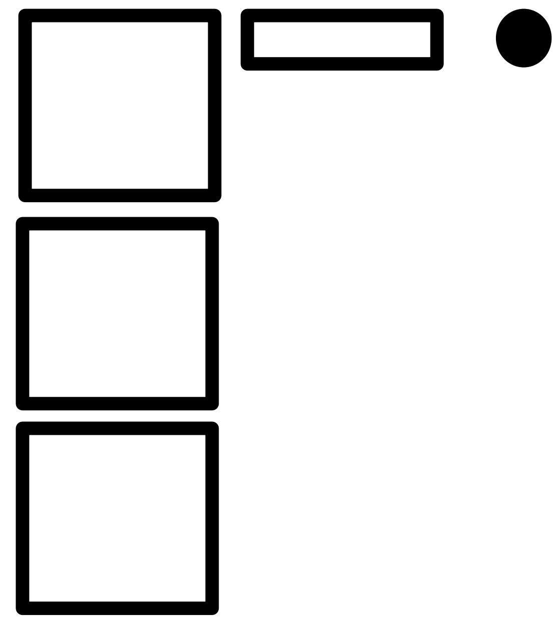
Combining Expressions: Three Column Notes

$$\left(1x^2 - 2x + 3\right) - \left(-2x^2 - 3x + 4\right) = 3x^2 + 1x - 1$$



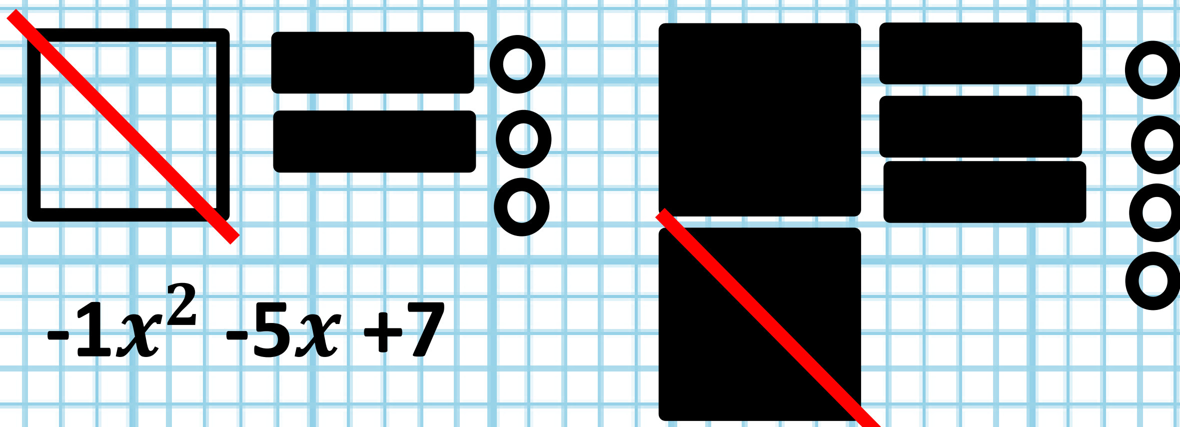
If you do subtraction, then make them change the shapes to the opposite before combining.





Next Level of Abstraction: Drawing the Tiles Around the Symbols

$(\ 1x^2 \quad -2x \quad +3\ ) + (\ -2x^2 \ -3x \quad +4\ )$



$-1x^2 \ -5x \ +7$

$(\boxed{11x^2}\boxed{+32x} +13\ ) + (\ \boxed{22x^2}\ \boxed{-34x}\ +14\ )$

$33x^2 \ -2x \ +27$

This is a good time to introduce larger coefficients.

THE HORIZONTAL METHOD IS GOOD FOR DRAWING SHAPES AROUND OR BELOW.

Make your squares, quadratic, large and your rectangles, linear, smaller, but with the same side length.

Constants get circles instead of the little squares that are more difficult to draw and easily confused with quadratics.

Final Level of Abstraction: Line Up Terms Vertically

$$(1x^2 \quad -2x \quad +3) - (-2x^2 \quad -3x \quad +4)$$

$$\begin{array}{rrr} +2x^2 & +3x & -4 \\ \hline 3x^2 & +1x & -1 \end{array}$$

Keep asking, "Can I simplify?"

THE VERTICAL METHOD IS GOOD FOR KEEPING LIKE TERMS TOGETHER AND REWRITING SUBTRACTION.

You can reference the shapes as you do it . You want students to follow the procedure of combining like terms based on what they learned about the areas they represent.

You can redraw them at the end to reinforce it.

Final Level of Abstraction: Line Up Terms Vertically

$$(\,1x^2 \quad -2x \quad +3\,) - (\,-2x^2 \; -3x \; +4\,)$$

$$\begin{array}{rrr} +2x^2 & +3x & -4 \\ \hline \boxed{3x^2} & \boxed{+1x} & -1 \end{array}$$

No, quadratic, linear & constant

THE VERTICAL METHOD IS GOOD FOR KEEPING LIKE TERMS TOGETHER AND REWRITING SUBTRACTION.

You can reference the shapes as you do it . You want students to follow the procedure of combining like terms based on what they learned about the areas they represent.

You can redraw them at the end to reinforce it.

Polynomial Multiplication: Area & Partial Products

David Mattoon
Meaning for Memory

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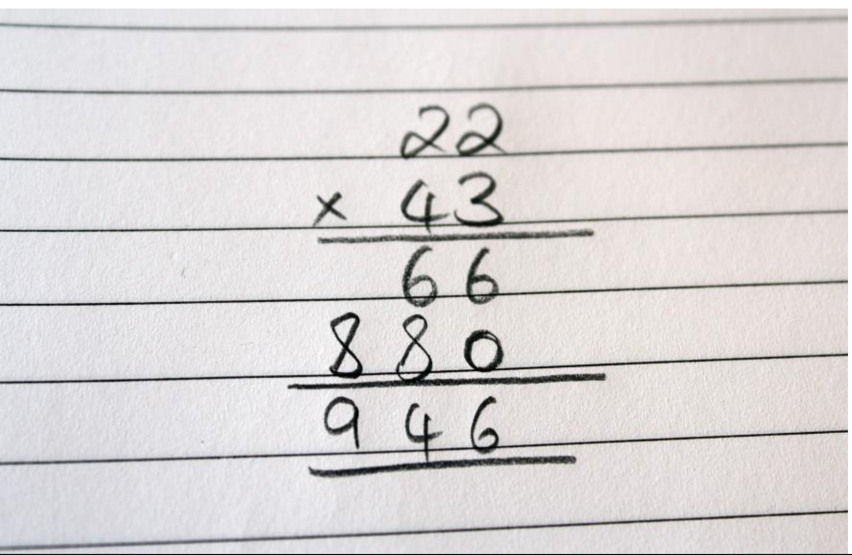
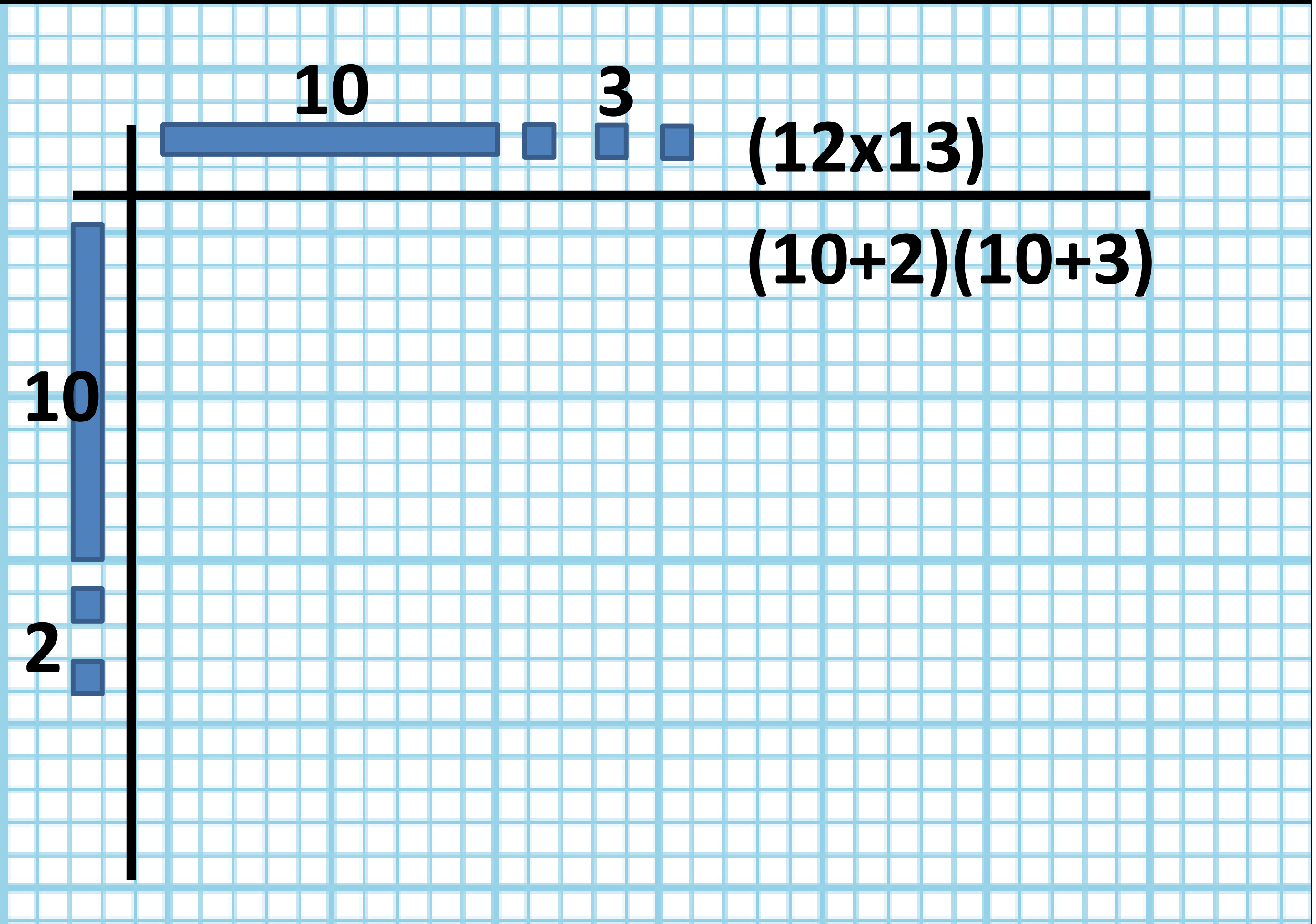
Lesson Introduction: Race the Calculator



22	20 + 2
x 43	x 40 + 3
800	800
80	80
60	60
6	6
946	946

Find the Product of 12 x 13 Using Base Ten Blocks

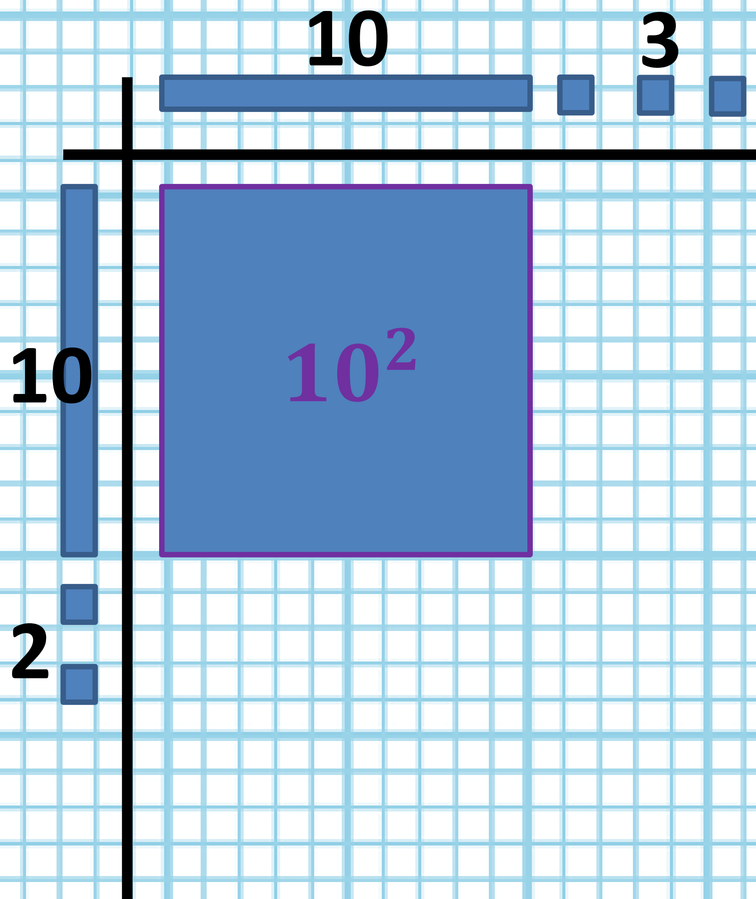


Start with base ten.

Why don't you follow along with me by drawing the two lines for the factors or the length of the sides and drawing base ten blocks.

Find the Product of 12 x 13 Using Base Ten Blocks

$$10 \cdot 10 = 10^2$$

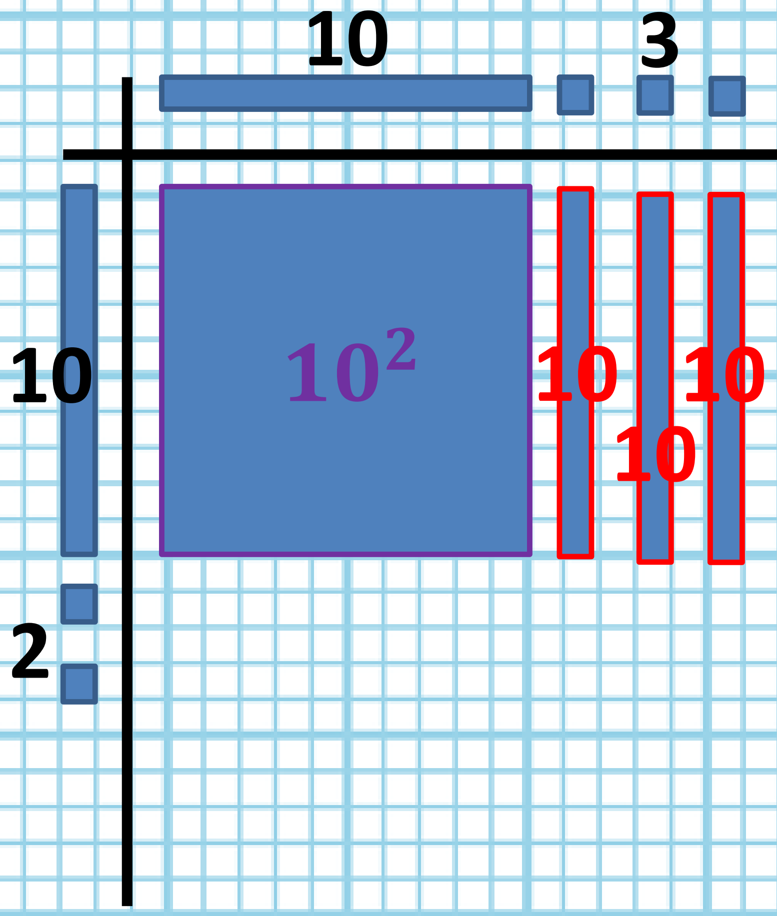


(12x13)

(10+2)(10+3)

$1(10^2)$

Find the Product of 12 x 13 Using Base Ten Blocks



(12x13)

(10+2)(10+3)

$1(10^2) + 3(10)$

$(10)(10)=$

10^2

$3(10)=$

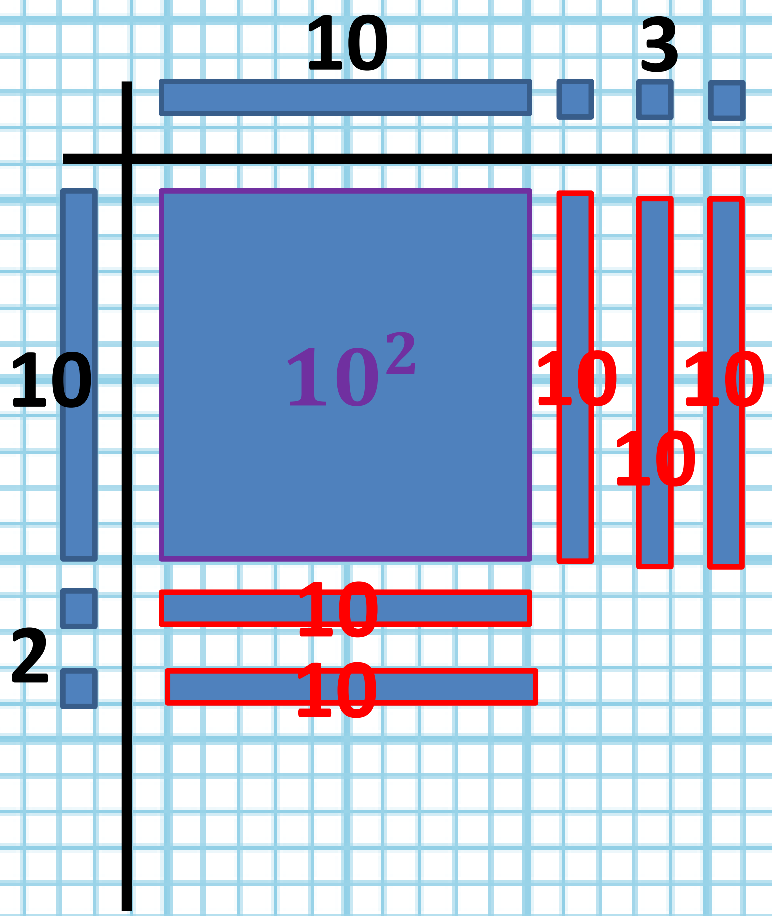
30

Find the Product of 12 x 13 Using Base Ten Blocks

$(10)(10) = 10^2$

$3(10) = 30$

$2(10) = 20$

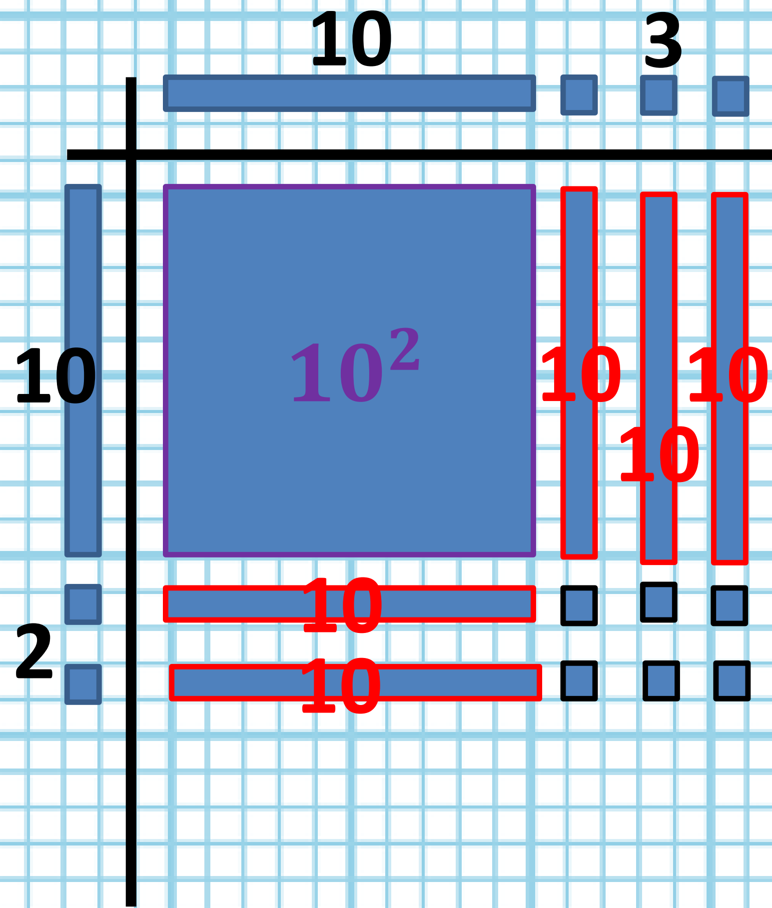


(12x13)

(10+2)(10+3)

$1(10^2) + 3(10) + 2(10)$

Find the Product of 12 x 13 Using Base Ten Blocks



(12x13)

(10+2)(10+3)

$1(10^2) + 3(10) + 2(10) + 6(1)$

(10)(10)

$=10^2$

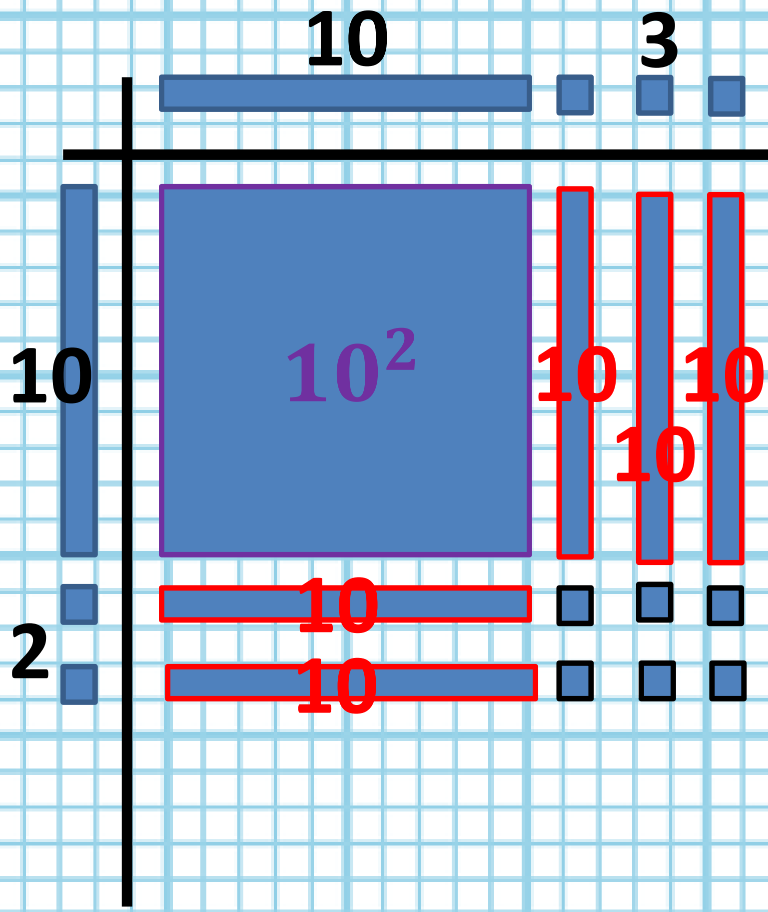
3(10) = 30

2(10) = 20

2(3) =6

David Mattoon Meaning for Memory

Find the Product of 12 x 13 Using Base Ten Blocks



(12×13)

$(10+2)(10+3)$

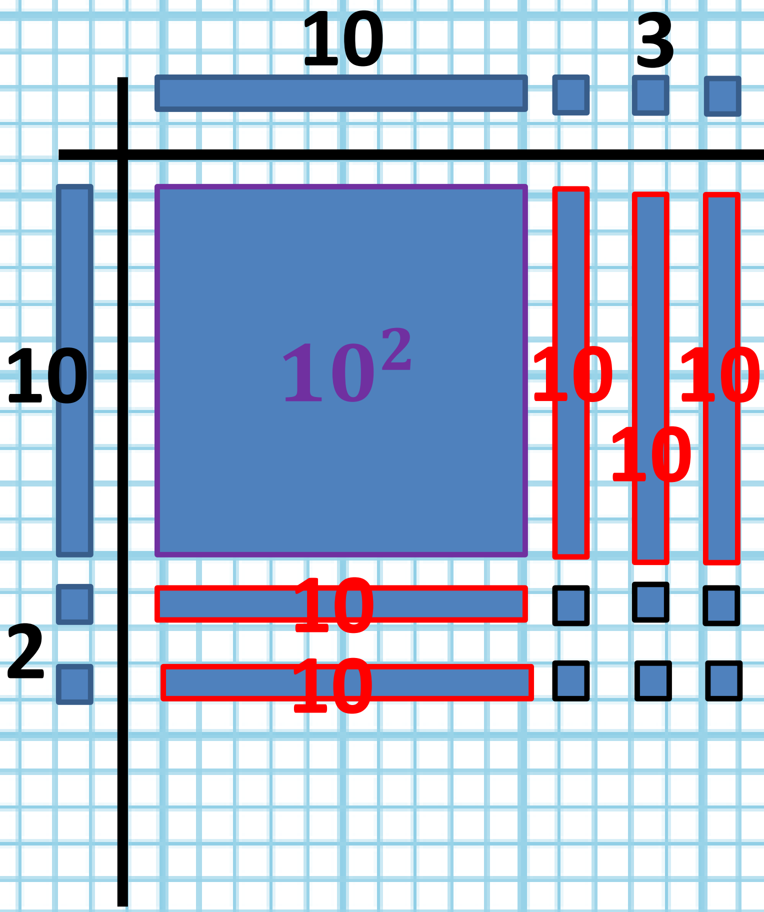
$1(10^2) + 3(10) + 2(10) + 6(1)$

$1(10^2) + 5(10) + 6(1)$

So combining the linear terms, the tens, we get 5(10).

Find the Product of 12 x 13 Using Base Ten Blocks

Combine the values of the terms to get 156.



(12x13)

(10+2)(10+3)

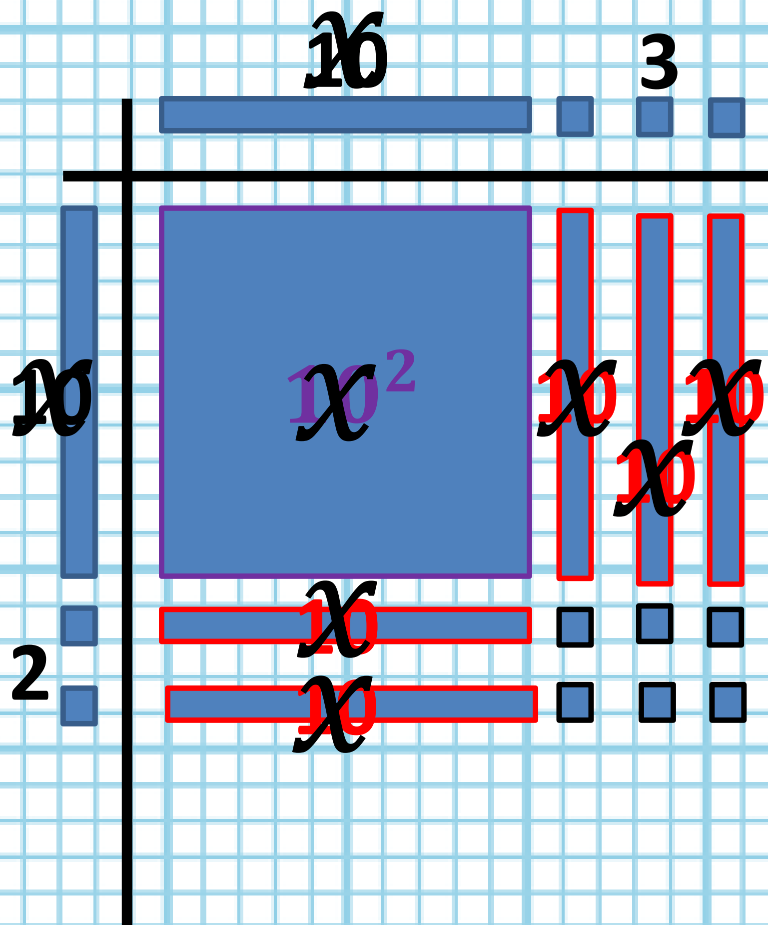
$1(10^2) + 3(10) + 2(10) + 6(1)$

$1(10^2) + 5(10) + 6(1)$

100 + 50 + 6

156

Transition to Algebra Tiles



$(x+2)(x+3)$

$1(x^2) + 3(x) + 2(x) + 6(1)$

$1(x^2) + 5(x) + 6(1)$

$x^2 + 5x + 6$

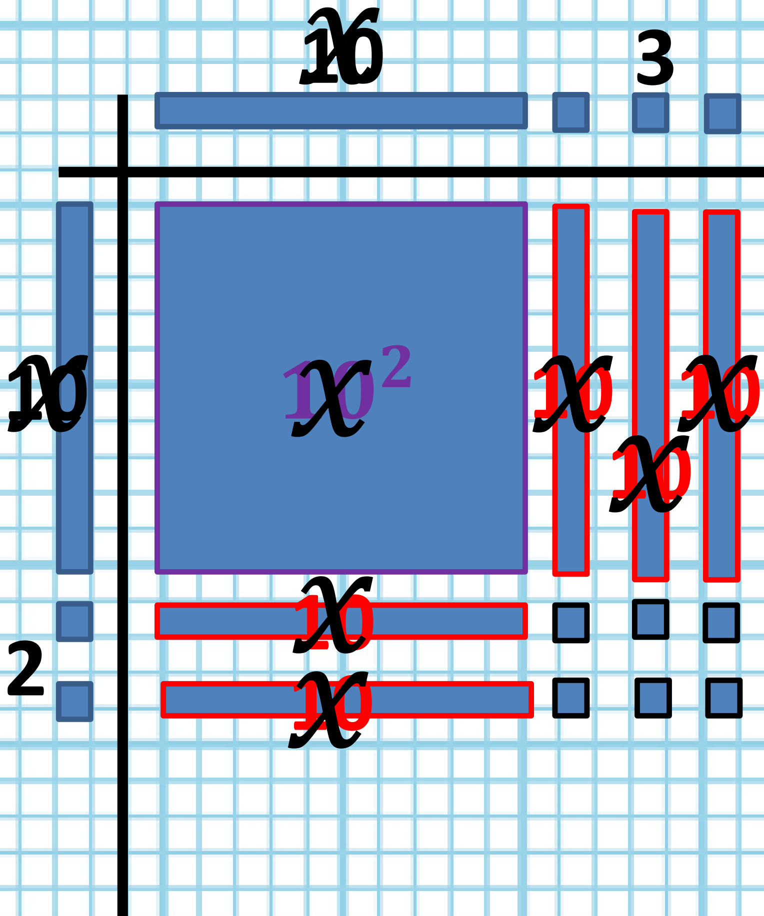
$1x^2 + 5x + 6$

I hate the number ten!

I want it to be any number.

Transition to Algebra Tiles

Everywhere you see a 10 replace it with x by erasing on your whiteboard



$(x+2)(x+3)$

$1(x^2) + 3(x) + 2(x) + 6(1)$

$1(x^2) + 5(x) + 6(1)$

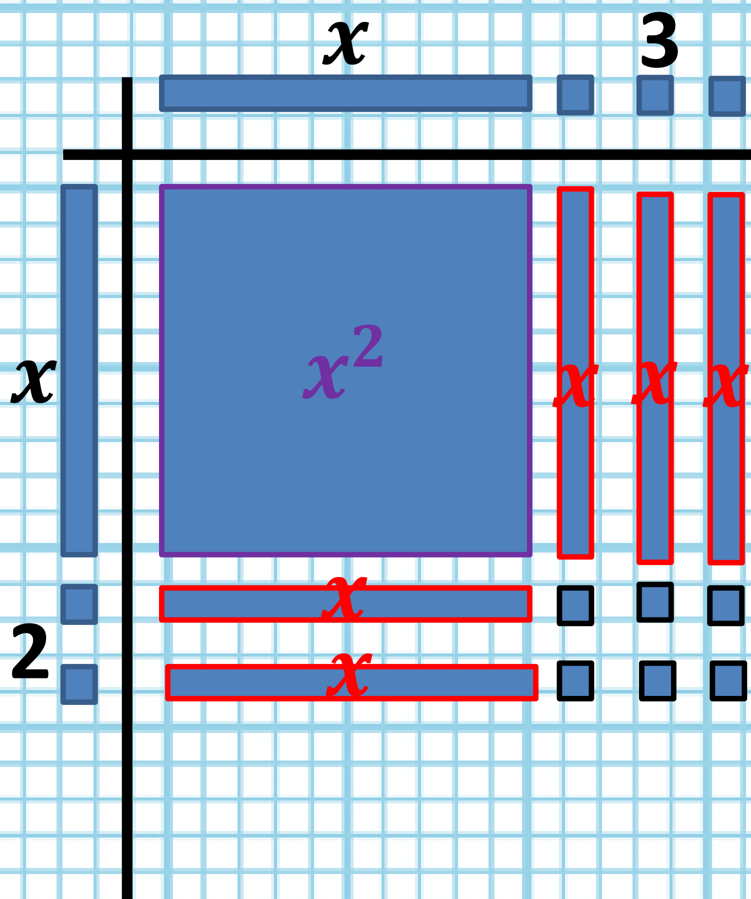
$x^2 + 5x + 6$

$1x^2 + 5x + 6$

Why can't you combine them all like in base ten?

Find the Product of (x + 2)(x + 3) Using Algebra Tiles

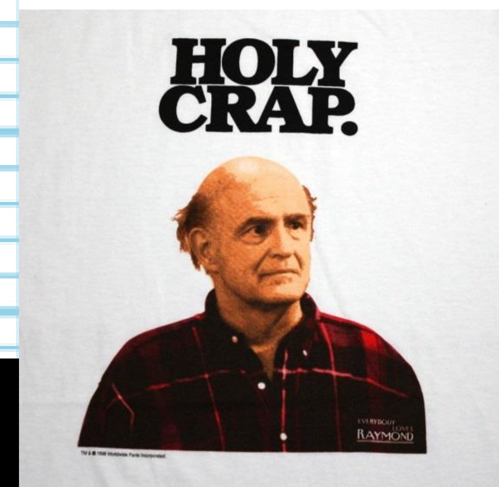
You are now finding the product in any base.



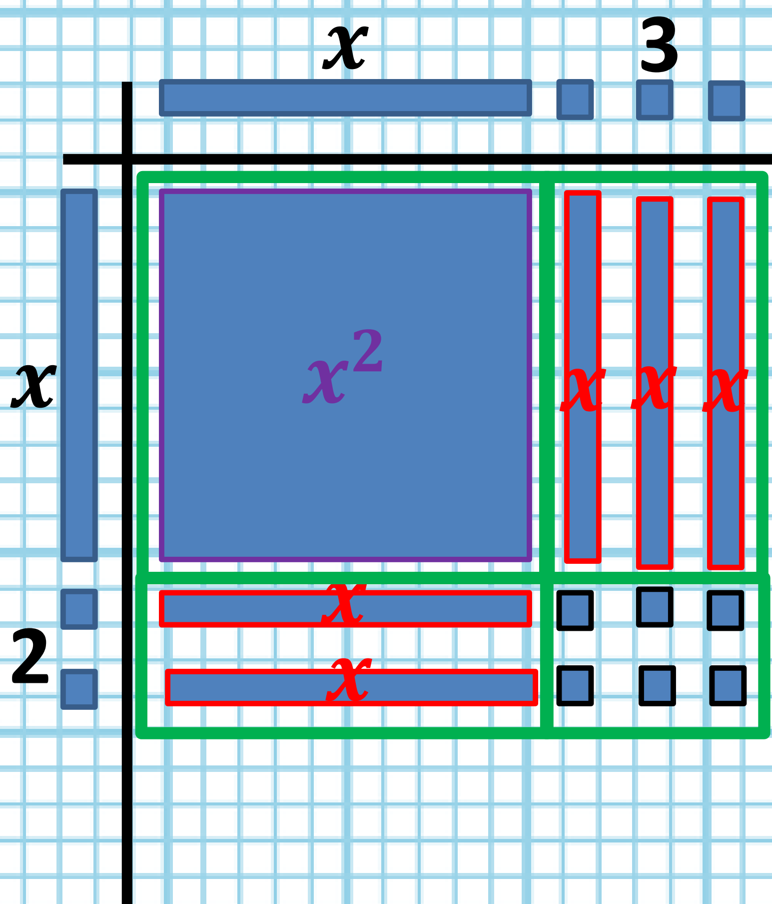
$(x+2)(x+3)$

$1(x^2) + 3(x) + 2(x) + 6(1)$

$1x^2 + 5x + 6$



Algebra Tiles: Working Toward the Next Level of Abstraction



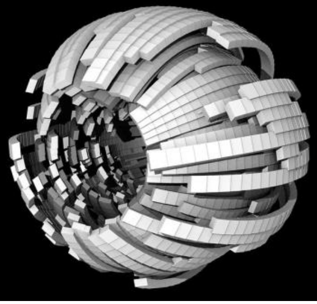
$(x+2)(x+3)$

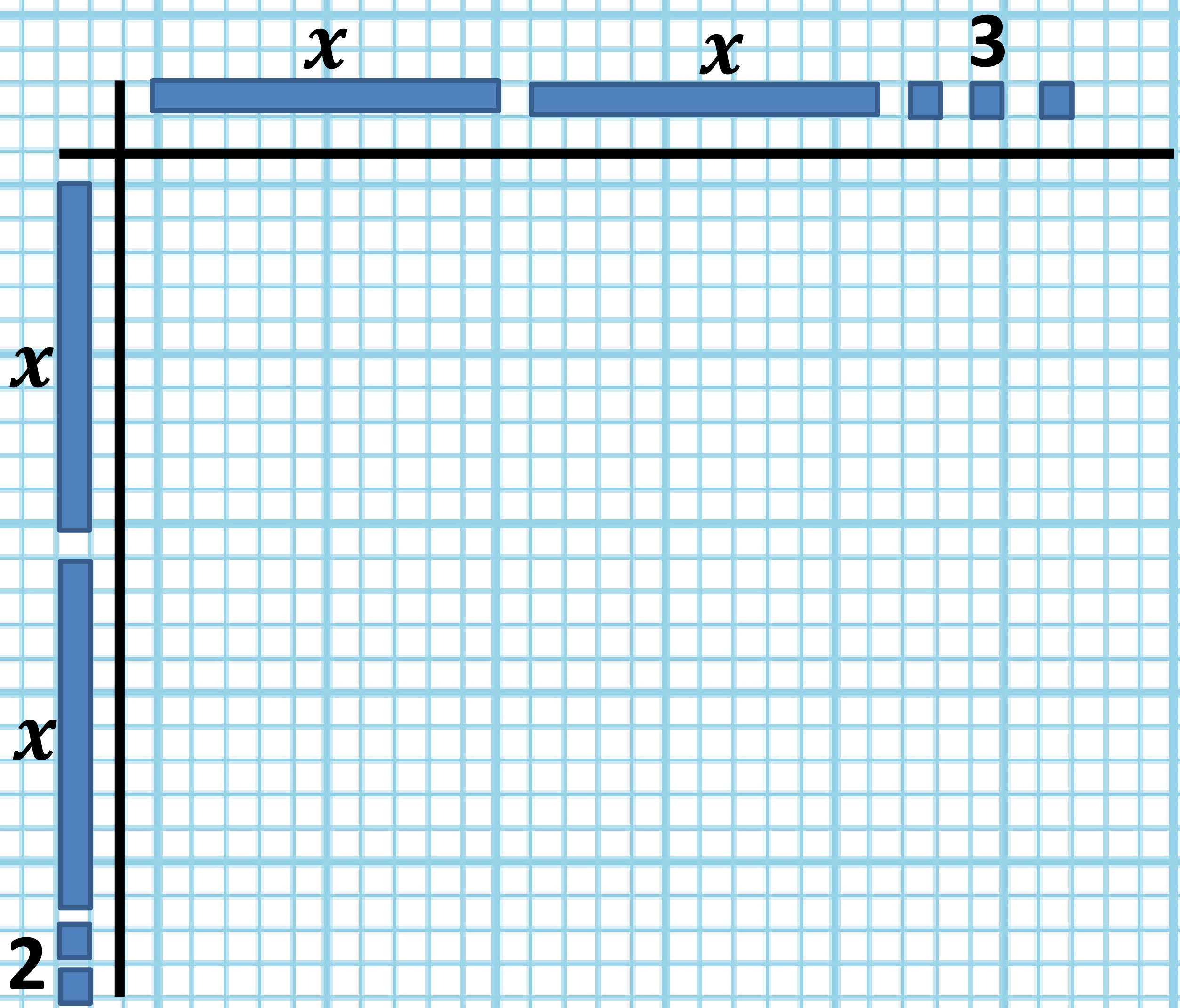
$1(x^2) + 3(x) + 2(x) + 6(1)$

$1x^2 + 5x + 6$

Notice the four regions of area… abstraction is coming…

Try One: Find the Product of (2x + 2)(2x + 3) Using Algebra Tiles

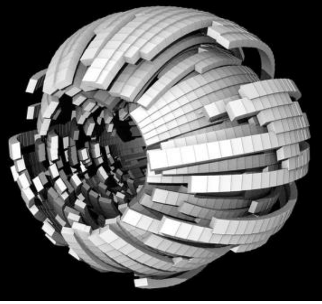


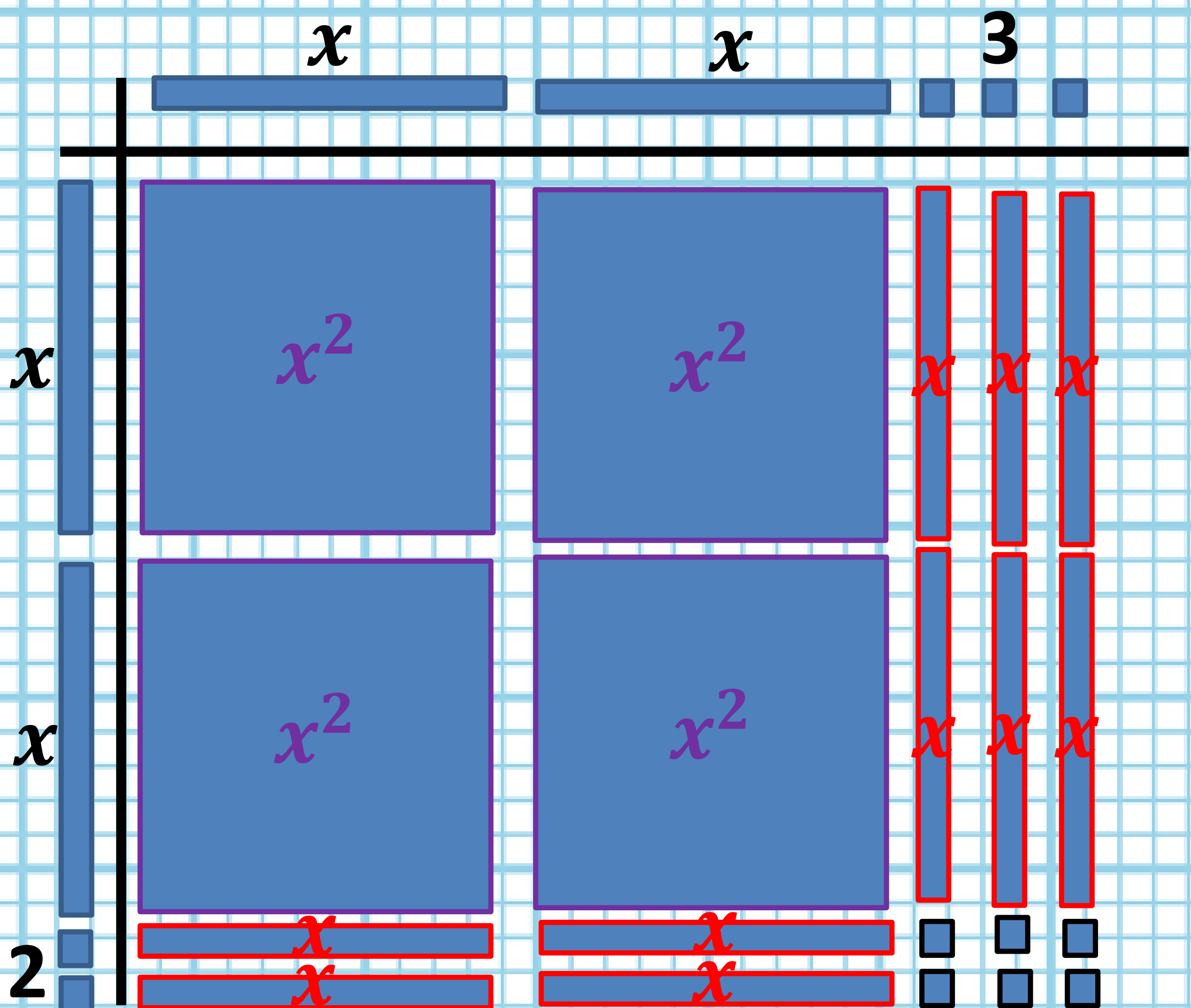


Use your whiteboard as a frame. Put the side lengths on the outside edge off the board and the product on the board.

Write your answer as a trinomial on your whiteboard, then check your answer with a partner.

Try One: Find the Product of (2x + 2)(2x + 3) Using Algebra Tiles





$(2x+2)(2x+3)$

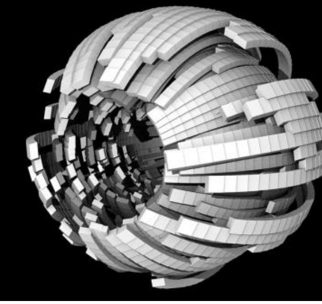
$4(x^2) + 6(x) + 4(x) + 6(1)$

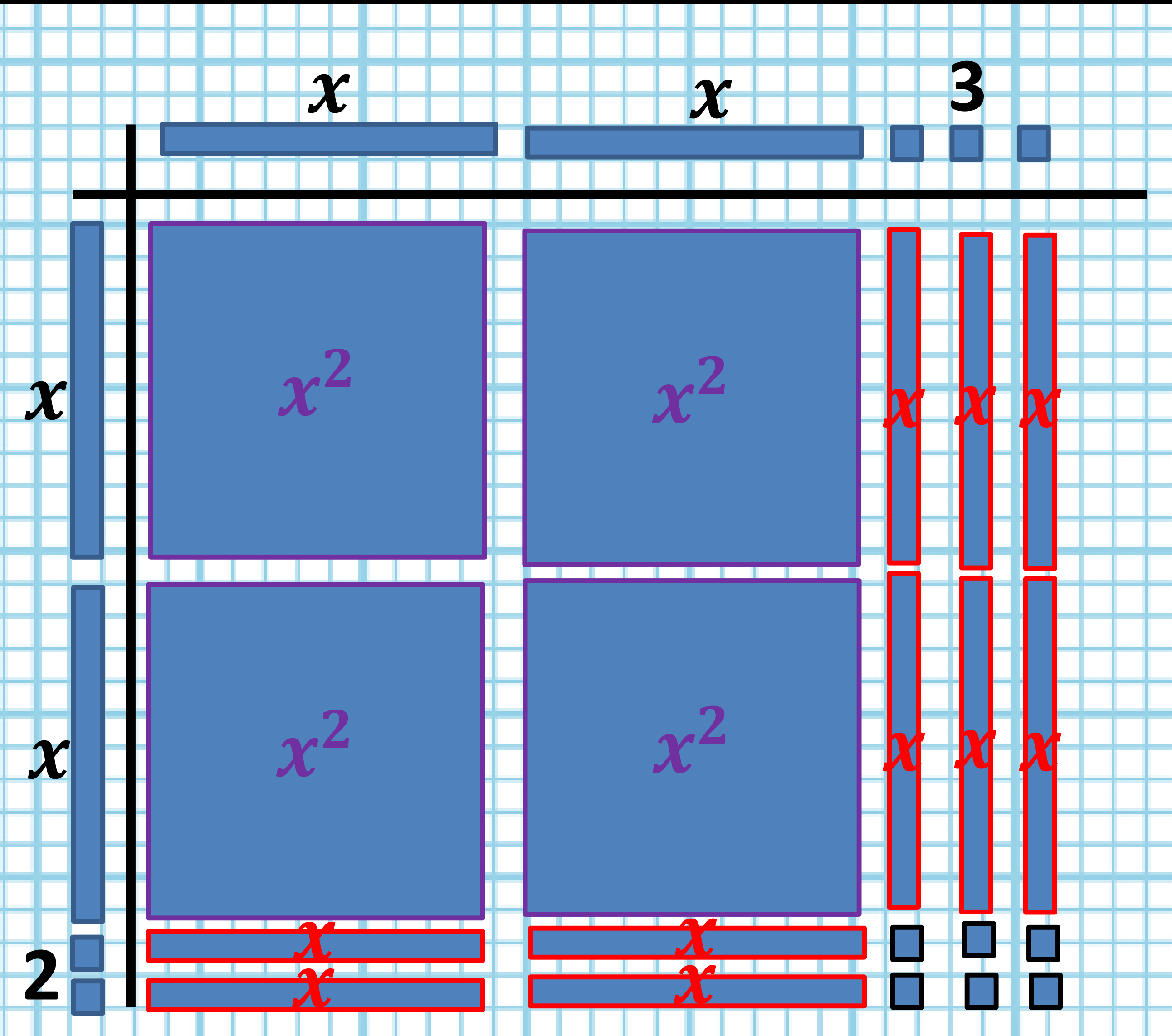
$4x^2 + 10x + 6$

If this had been base ten, then what problem would this have been, the two side lengths multiplied ?

If the base had been ten, then what would the product, or the area, have been?

Try One: Find the Product of (2x + 2)(2x + 3) Using Algebra Tiles





$(2x+2)(2x+3)$

$4(x^2) + 6(x) + 4(x) + 6(1)$

$4x^2 + 10x + 6$

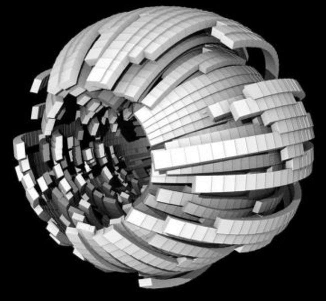
If this had been base ten, then what problem would this have been, the two side lengths multiplied ?

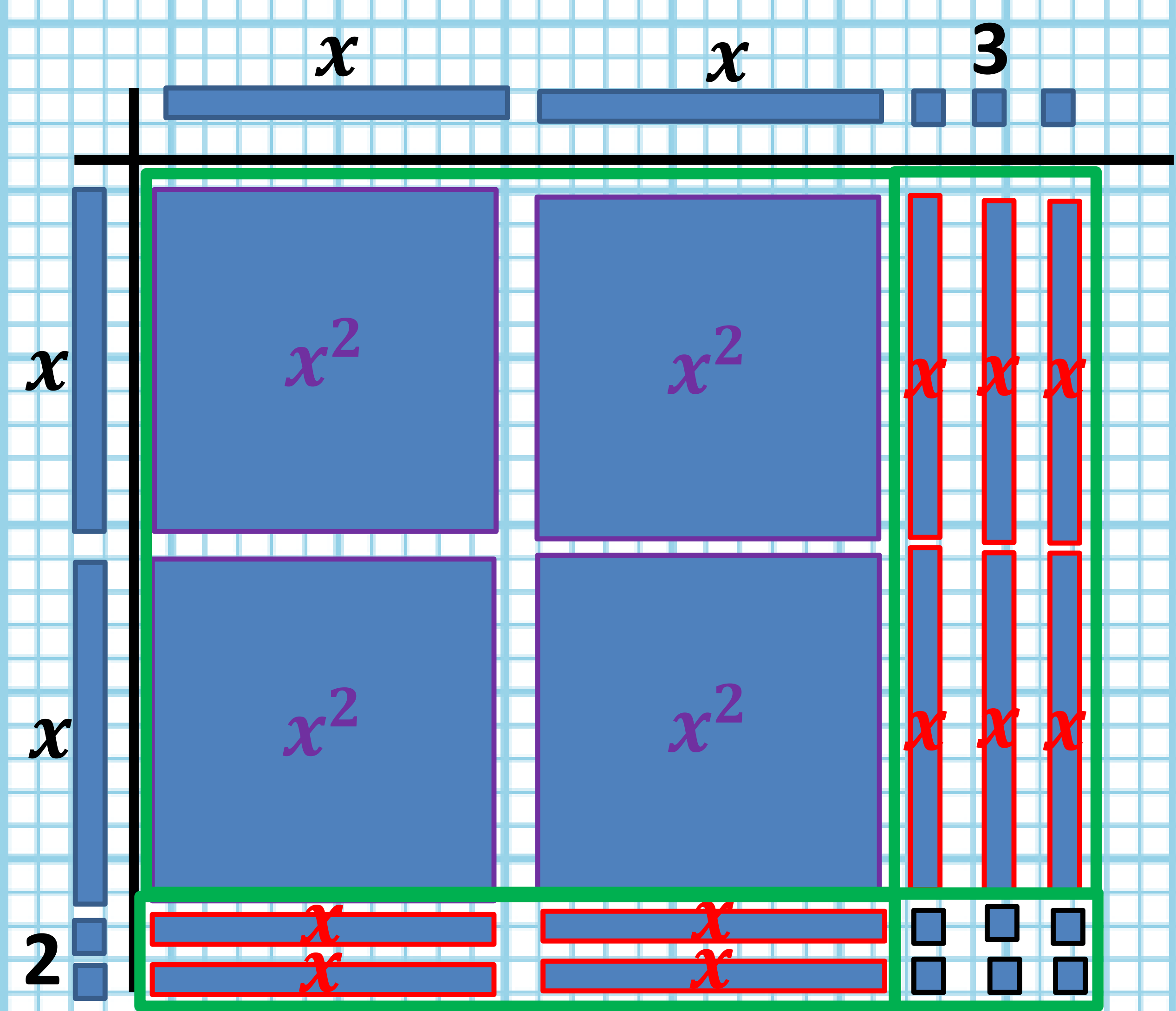
(22)(23)

If the base had been ten, then what would the product, or the area, have been?

506

Try One: Find the Product of (2x + 2)(2x + 3) Using Algebra Tiles





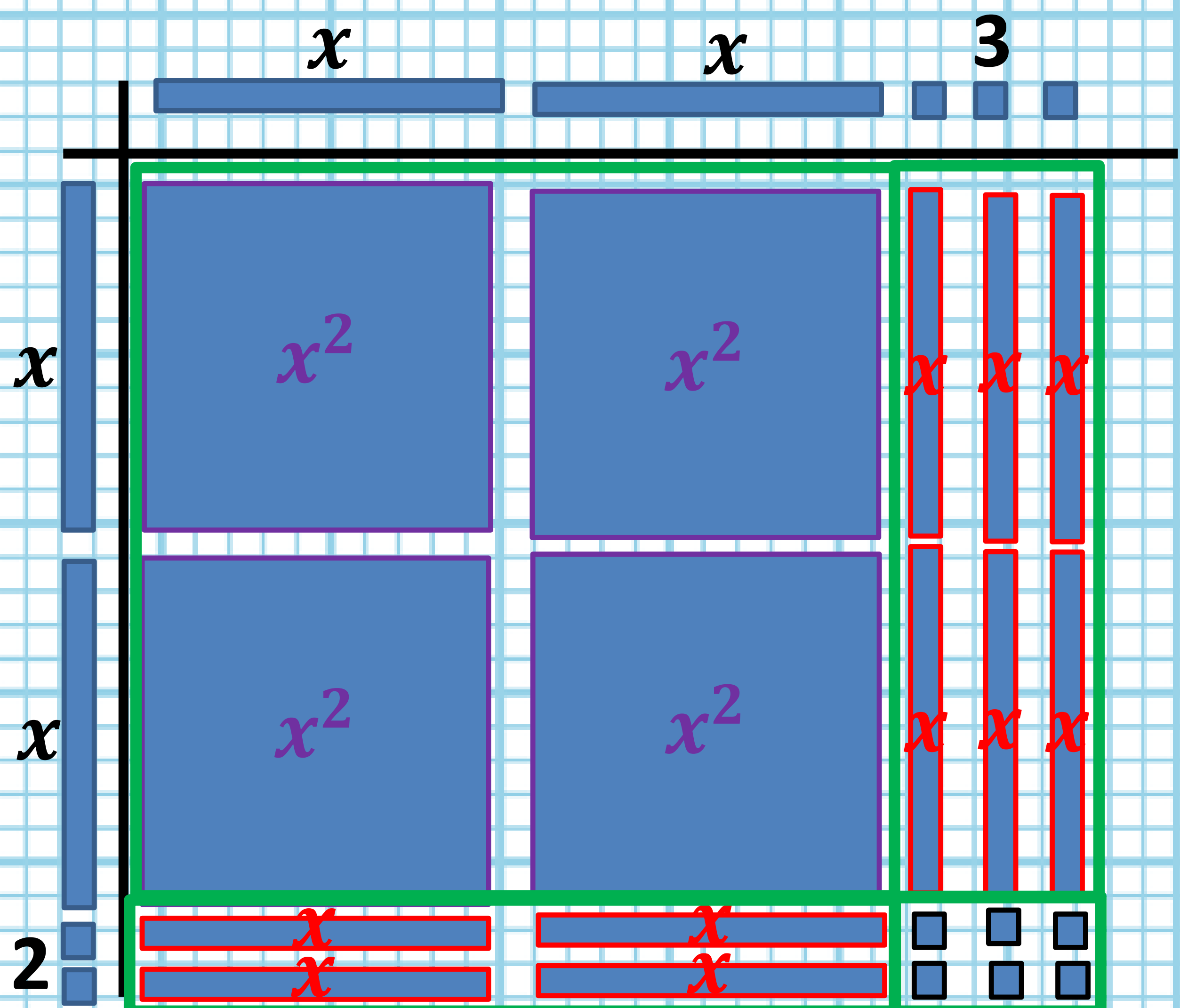
Notice the four regions of area… abstraction is coming…

$(2x+2)(2x+3)$

$4(x^2) + 6(x) + 4(x) + 6(1)$

$4x^2 + 10x + 6$

The Effect of Standard Form



From the two examples we have done:

What do you notice about the quadratic (square) term in the picture?

It is always on the

______ ______.

What do you notice about the linear terms in the picture?

They are always on the ______.

What do you notice about the constant term in the picture?

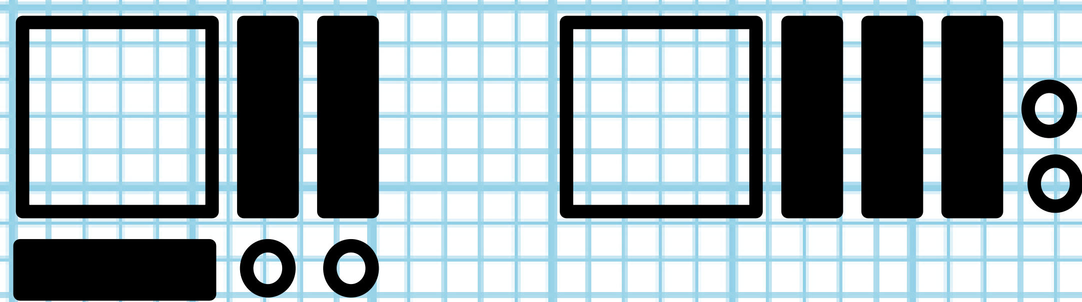
It is always on the

______ ______.

Word Bank:

Top Bottom
Left Right
Diagonal

Next Level of Abstraction: Draw the Shapes



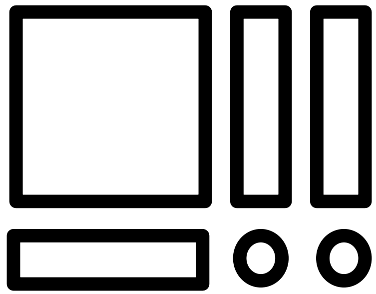
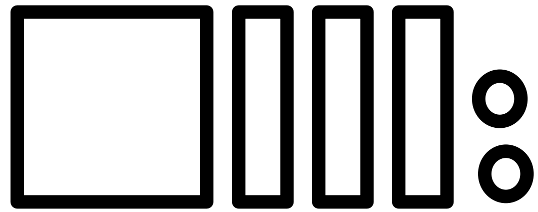
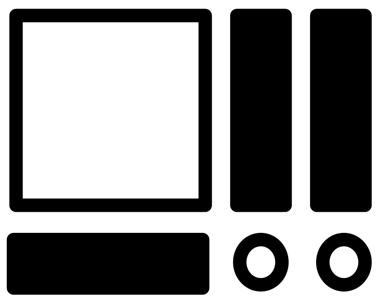
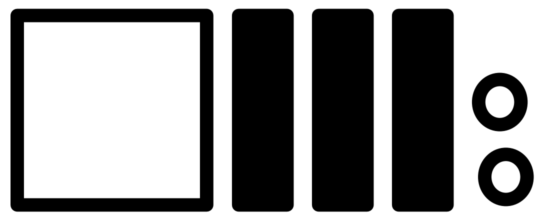
Open terms are positive and closed terms are negative.

Referring to them as white being positive and black being negative is not a good idea.

Fold Your Paper into Four Columns

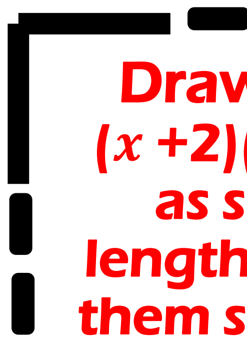
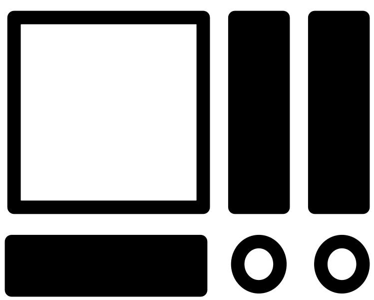
Factored Expression	Factored Pictorial	Unfactored Pictorial	Unfactored Expression

All Four Representations to Build Understanding

Factored Expression	Factored Pictorial	Unfactored Pictorial	Unfactored Expression
$(x+2)(x+1)$			x^2+3x+2
$(x-2)(x-1)$			x^2-3x+2

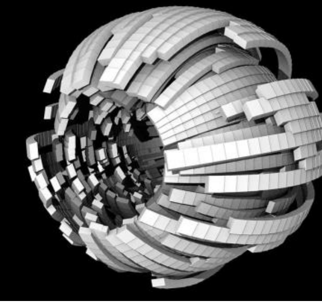
I included a negative term here as an example if you choose to go there before higher levels of abstraction.

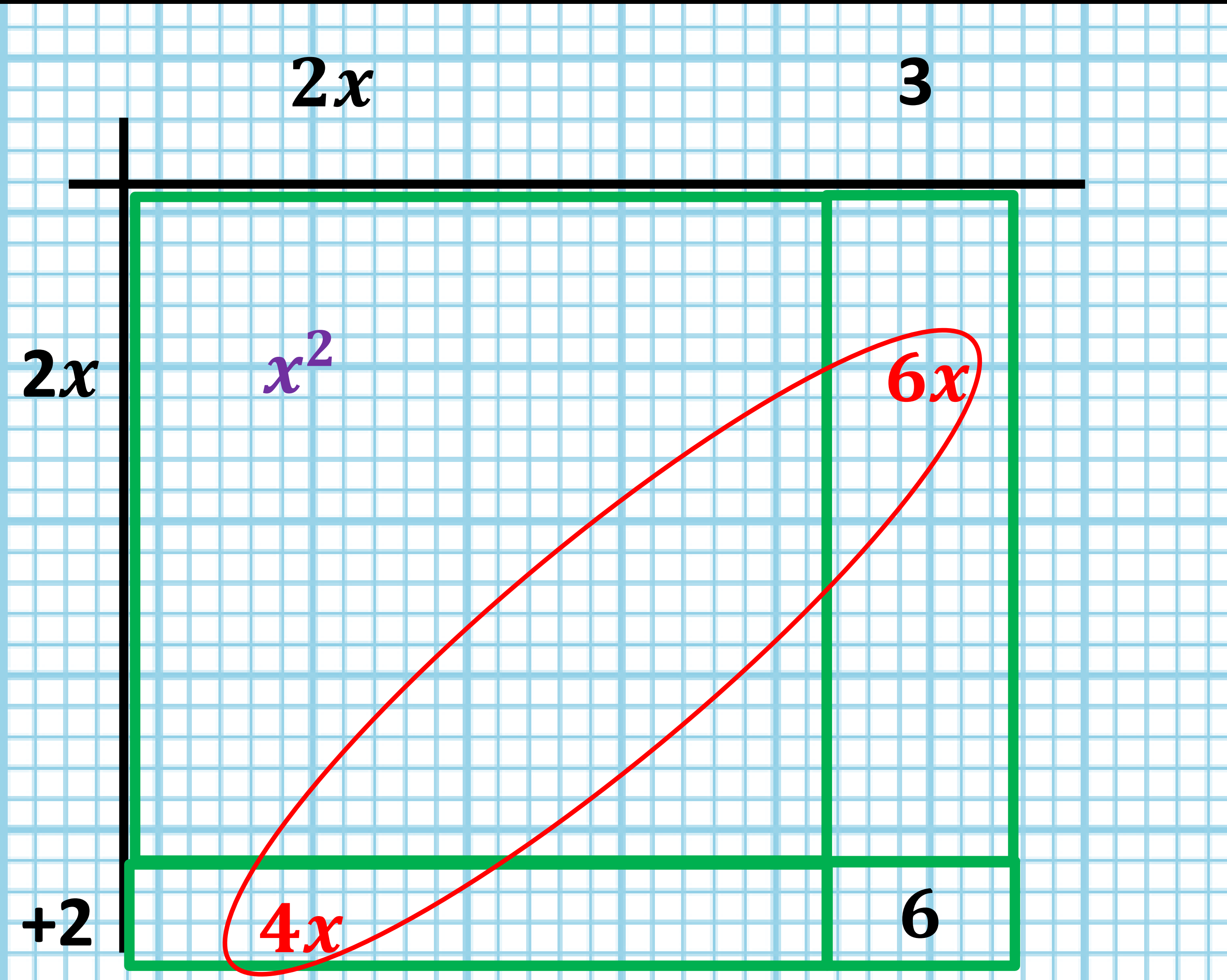
Move Fluidly between Representations

Factored Expression	Factored Pictorial	Unfactored Pictorial	Unfactored Expression
$(x+2)(x+1)$	 Drawing $(x+2)(x+1)$ as side lengths gets them started		
			

Once they have practiced, you can give them a single expression and ask for the others.

Next Level of Abstraction Area Representation without the Tiles (to scale)





I rarely use this abstraction step; however, I might show it once to students.

Box Method?

The terms represent the tiles or the area of the rectangle they are within, which needs to be reinforced. The are not merely entries in a table or box.

Still $(2x+2)(2x+3)$

$4(x^2) + 6(x) + 4(x) + 6(1)$

$4x^2 + 10x + 6$

Emphasize combining like terms on the diagonal

Table, Chart or Boxes?

12 X 12 Multiplication Table

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

Tell someone next you what you think.

Table, Chart or Boxes?

12 X 12 Multiplication Table

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

Trick Question:

None of the above, AREA!

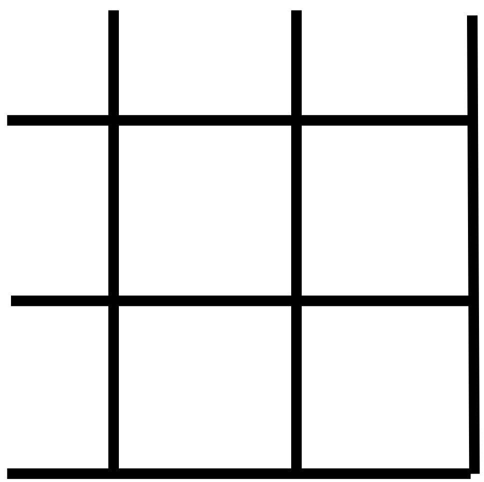
Check Out (11)(12) or

(10+1)(10+2)

100+30+2

132

Next Level of Abstraction: Box Method



The box method is a level of abstraction of the area method. Taught without the algebra tiles it means no more than numbers in a box like a multiplication table.

Algebra tiles should be done first so when students use the Xbox they visualize the algebra tiles. Thus, avoiding the common mistake of FL instead of FOIL, which by the way is next to useless besides connecting to parents. FOIL only works for binomial multiplication. The distributive property is what we are working toward.

Next Level of Abstraction Area Representation without the Tiles (not to scale)

	$2x$	$+3$
$2x$		
$+2$		

$(2x+2)(2x+3)$

	$3x$	-4
x		
-3		

$(3x-4)(x-3)$

This is a good time to bring back negatives if you have skipped them for the more conceptual pieces.

By the way, negatives are effective for base 10 and mental multiplication like (50-1)(50-1) instead of (49)(49).

I like to extend the lines outside of the large, exterior rectangle to help students see negative signs, and it makes them resemble the tick marks indicating length.

Next Level of Abstraction Area Representation without the Tiles (not to scale)

	$2x$	$+3$
$2x$	$4x^2$	$6x$
$+2$	$4x$	6

$(2x+2)(2x+3)$

$4x^2 + 6x + 4x + 6$

$4x^2 + 10x + 6$

	$3x$	-4
x	$3x^2$	$-4x$
-3	$-9x$	12

$(3x-4)(x-3)$

$3x^2 - 4x - 9x + 12$

$3x^2 - 13x + 6$

You are using the side lengths to find the area of each rectangle (partial products). Emphasize this with students.

Continue to emphasize combining like terms on the diagonal.

Try One: Multiplying Two Trinomials

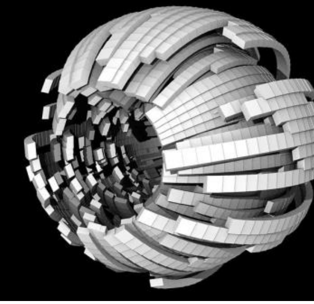
	$6x^2$	$+5x$	$+4$
$3x^2$			
$+2x$			
$+1$			

$$(6x^2 + 5x + 4)(3x^2 + 2x + 1)$$

On your whiteboard or notes, find the product.

What do you notice about the like terms?

Order In… Order Out



$27x^3$

	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$
$2x$	$12x^3$	$10x^2$	$8x$
$+1$	$6x^2$	$5x$	4

$28x^2$

$13x$

$(6x^2 + 5x + 4)(3x^2 + 2x + 1)$

$18x^4 + 12x^3 + 15x^3 + 6x^2 + 10x^2 + 12x^2 + 5x + 8x + 4$

$18x^4 + 27x^3 + 28x^2 + 13x + 4$

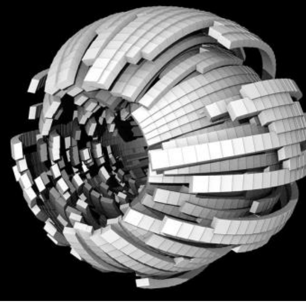
When you fill out the lengths using standard form, then the like terms are on the diagonals.

Depending on the polynomial this is not always true, but when it is it really helps avoid mistakes with mislabeling, especially for more complex problems.

Do a problem like $(6x^2 + 5x + 4)(3x^2 + 1)$ **to show students that it is not always the case.**

Time: I don’t have it.

Time Gained



	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$

$(3x^2)(6x^2 + 5x + 4)$

$18x^4 + 15x^3 + 12x^2$

Ask, "how many terms in the first factor?" and "How many terms in the second factor?' in order to draw the correct "box."

	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$
$2x$	$12x^3$	$10x^2$	$8x$

$27x^3$ $22x^2$

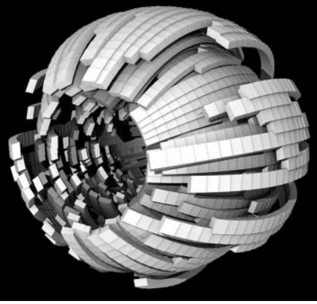
$(3x^2 + 2x)(6x^2 + 5x + 4)$

$18x^4 + 27x^3 + 22x^2 + 8x$

Time spent on algebra tiles is recouped by a single method used for multiple problem types.

More time is recouped by less time reviewing as there is only one method to remember in the first place or one method to review with students.

Remediation & SpEd



$27x^3$

	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$
$2x$	$12x^3$	$10x^2$	$8x$
$+1$	$6x^2$	$5x$	4

$28x^2$

$13x$

$(6x^2 + 5x+4)(3x^2+2x+1)$

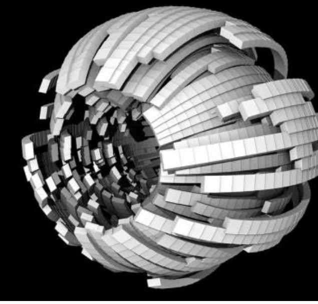
$18x^4 + 12x^3+15x^3+ 6x^2 + 10x^2 + 12x^2 + 5x + 8x +4$

$18x^4 + 27x^3+ 28x^2 + 13x +4$

Need to review math facts? Why not teach polynomials? Multiplication, addition, and subtraction through negatives.

Division? Give them the product and a factor and ask for the other factor, length of the unknown side.

Final Level of Abstraction
The Distributive Property



	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$

$(3x^2)(6x^2 + 5x+4)$

$18x^4 +15x^3+ 12x^2$

Top Row

$18x^4 +15x^3+12x^2$

$12x^3+10x^2 +8x$

Bottom Row

	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$
$2x$	$12x^3$	$10x^2$	$8x$

$(3x^2 + 2x)(6x^2 + 5x +4)$

$18x^4 +27x^3+ 22x^2+ 8x$

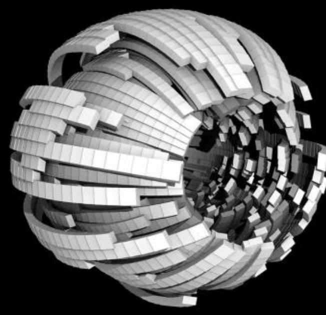
Students should be able visualize or use their experience with the area representation and/or the box to distribute correctly.

Line up like terms vertically like before to get the result in the second example.

Polynomial Factoring: Making Rectangles & Finding Side Lengths

California Mathematics Council
Southern Section

David Mattoon
Meaning for Memory



Using *only* a guess and check method solve for x:

$$2x^2 + 7x + 6 = 0$$

(T-charts are OK to keep track of your guesses.)

Why are we even factoring?

Did you feel the pain?

You want them to feel the pain so they appreciate the Zero Product Property

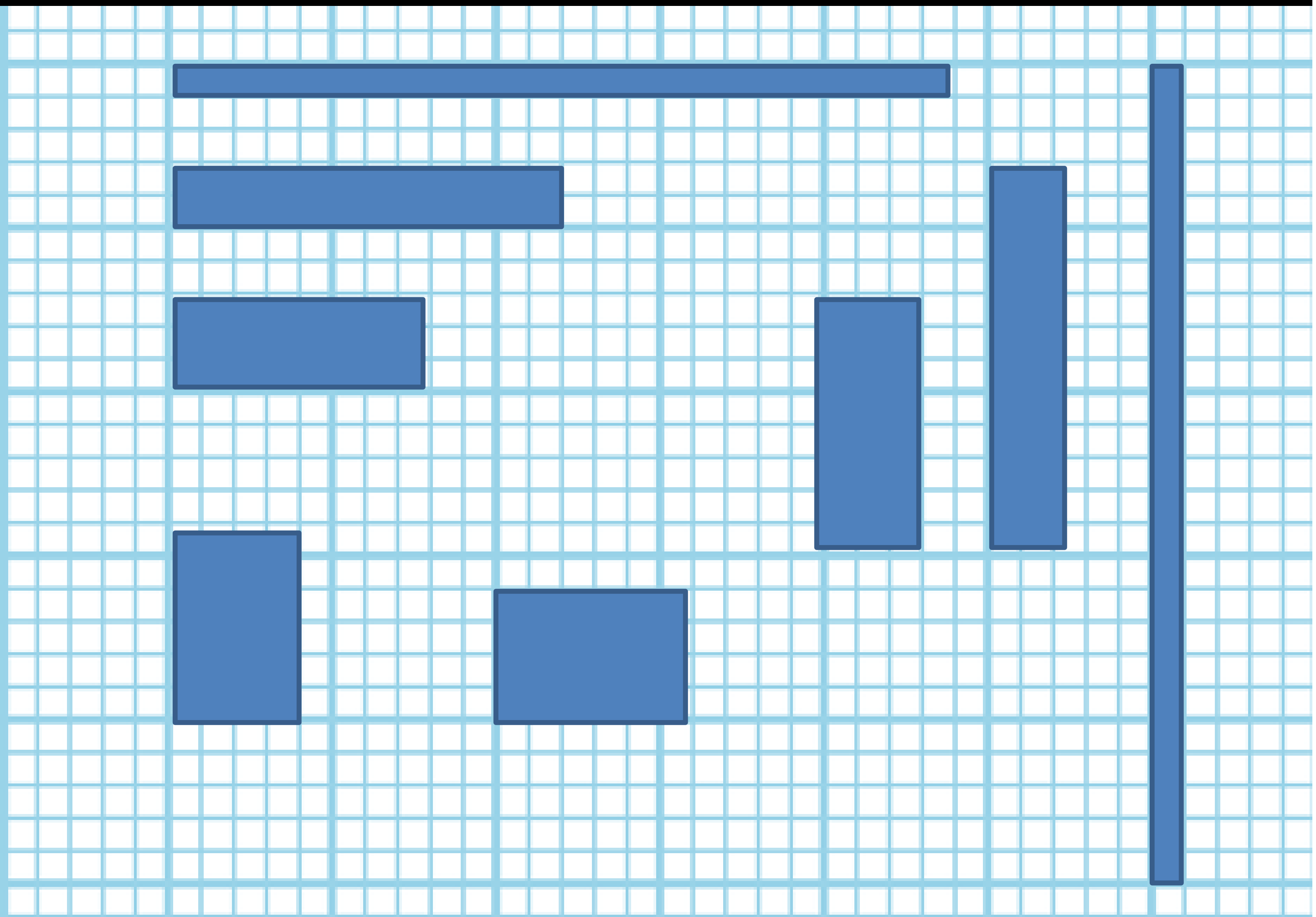
$(2x+3)(x+2) = 0$

But how do you get there?

Lesson Introduction: Make Rectangles

On the gridded side of your whiteboard, make as many rectangles as you can with an area of 24 square units.

Lesson Introduction: Make Rectangles



1 x 24

24 x 1

2 x 12

12 x 2

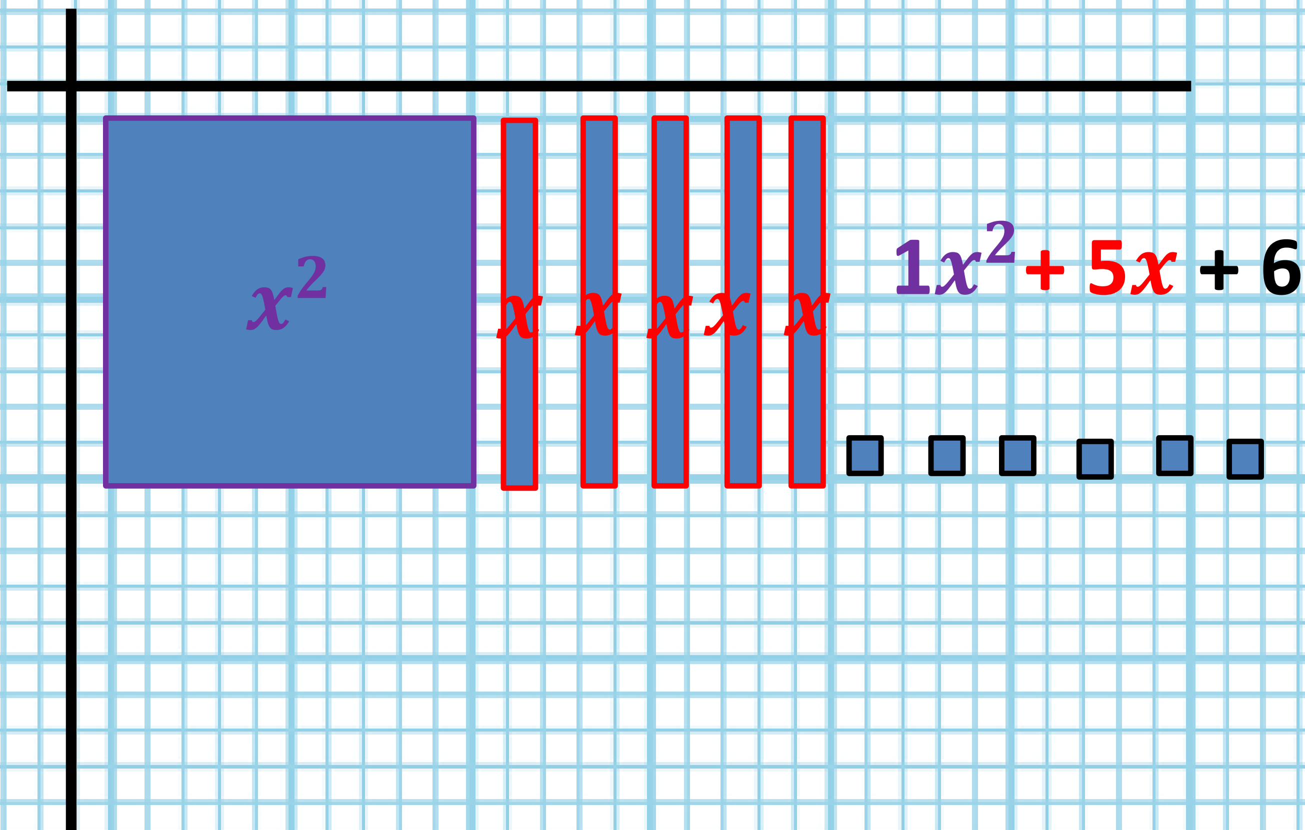
3 x 8

8 x 3

4 x 6

6 x 4

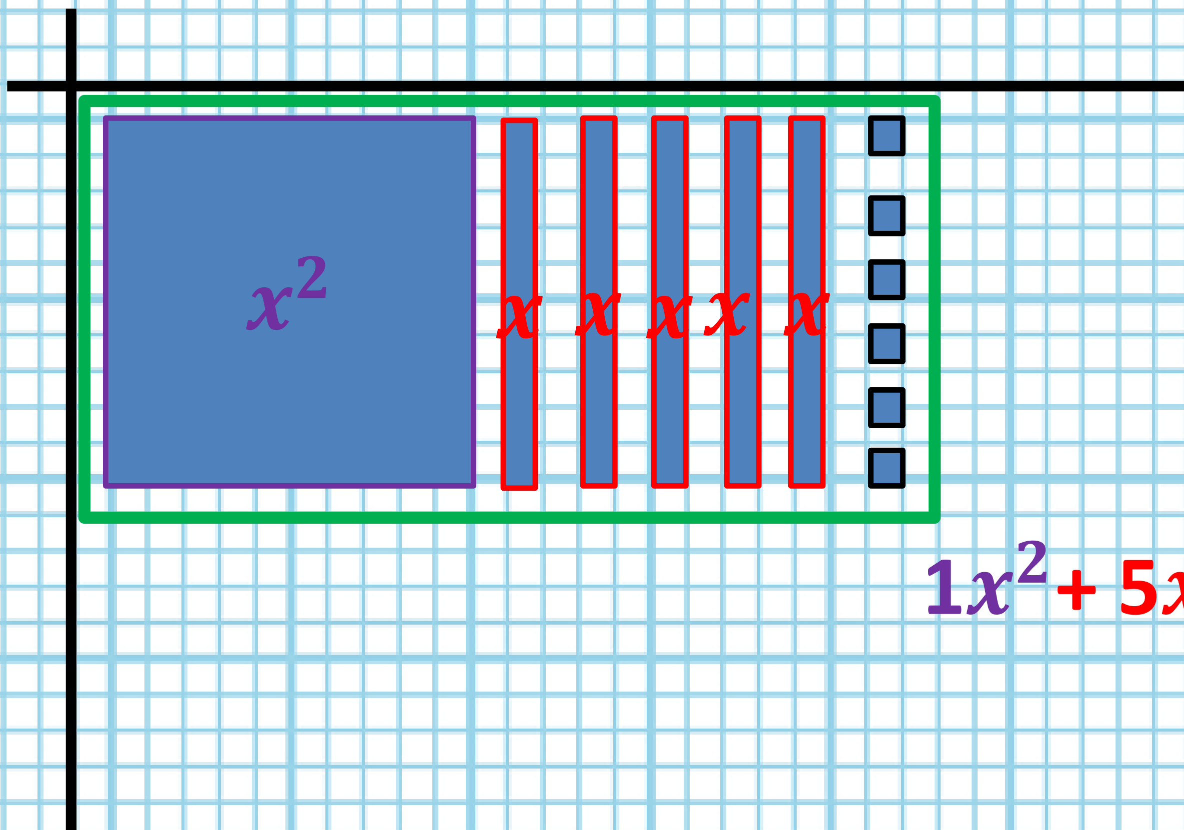
Try One: Factor $1x^2 + 5x + 6$ using Algebra Tiles



Draw the frame on your whiteboard leaving room to write in side lengths later and then factor $1x^2 + 5x + 6$.

Remember when you multiplied polynomials and make the area taken up by the shapes into a rectangle.

Try One: Factor $1x^2 + 5x + 6$ using Algebra Tiles



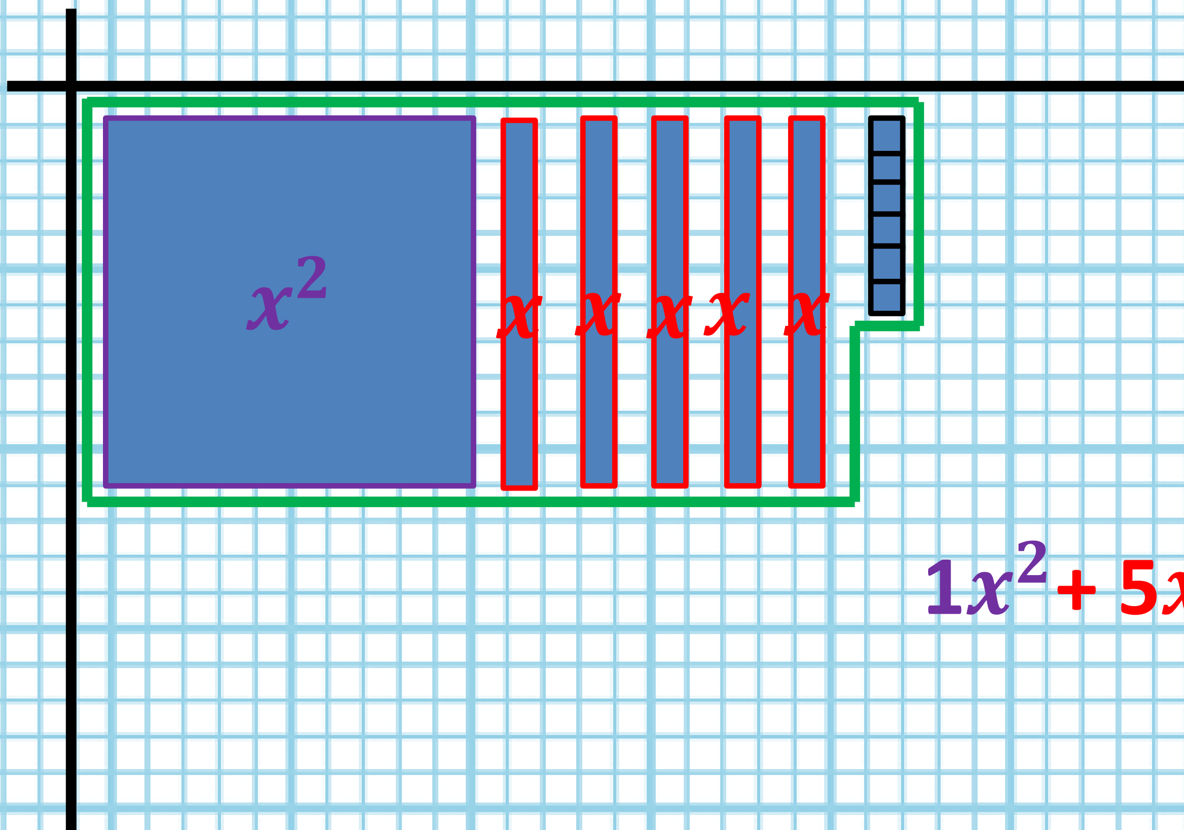
$1x^2 + 5x + 6$

Does this work?

No, you have not filled in the complete area in the last column.

This is really…

Try One: Factor $1x^2 + 5x + 6$ using Algebra Tiles



$1x^2 + 5x + 6$

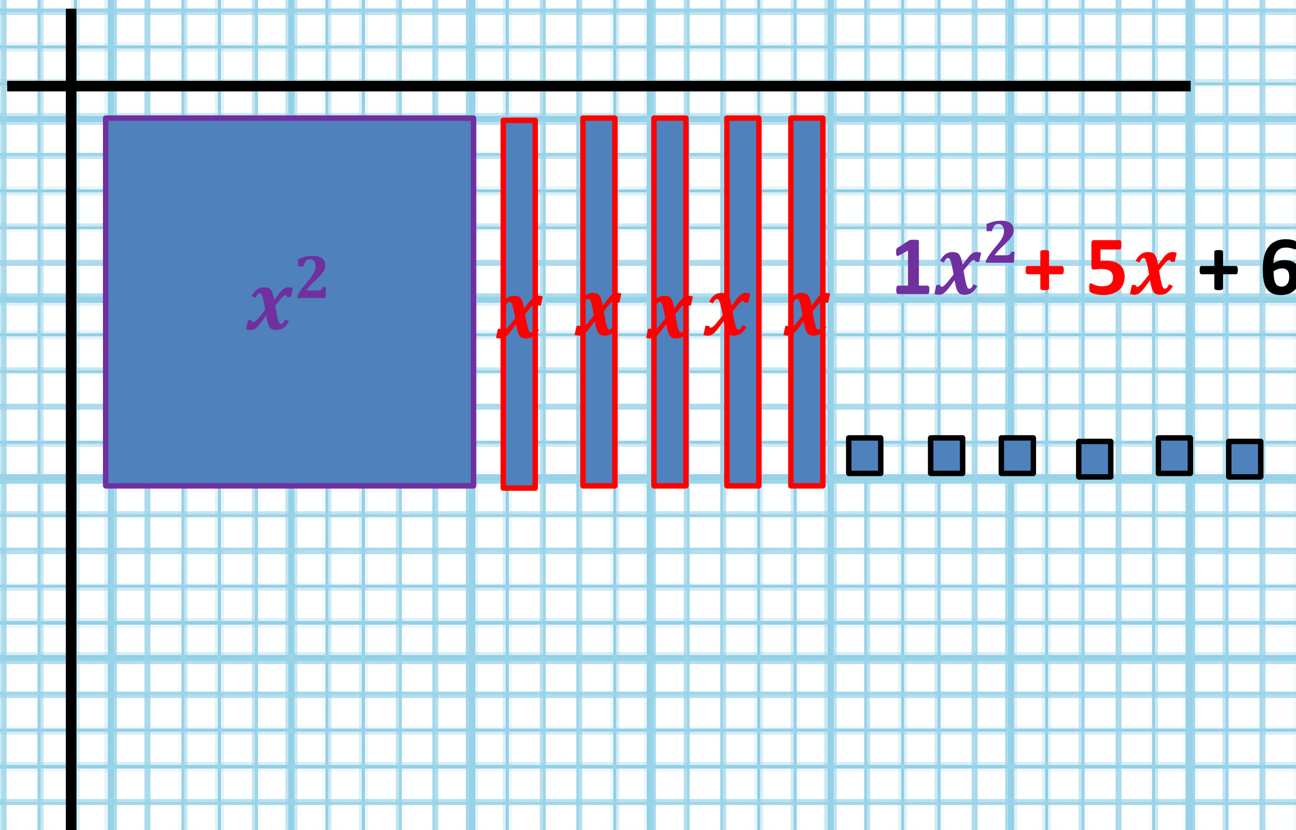
Does this work?

No, you have not filled in a complete area in the last column.

This is really, this.

This is not a rectangle.

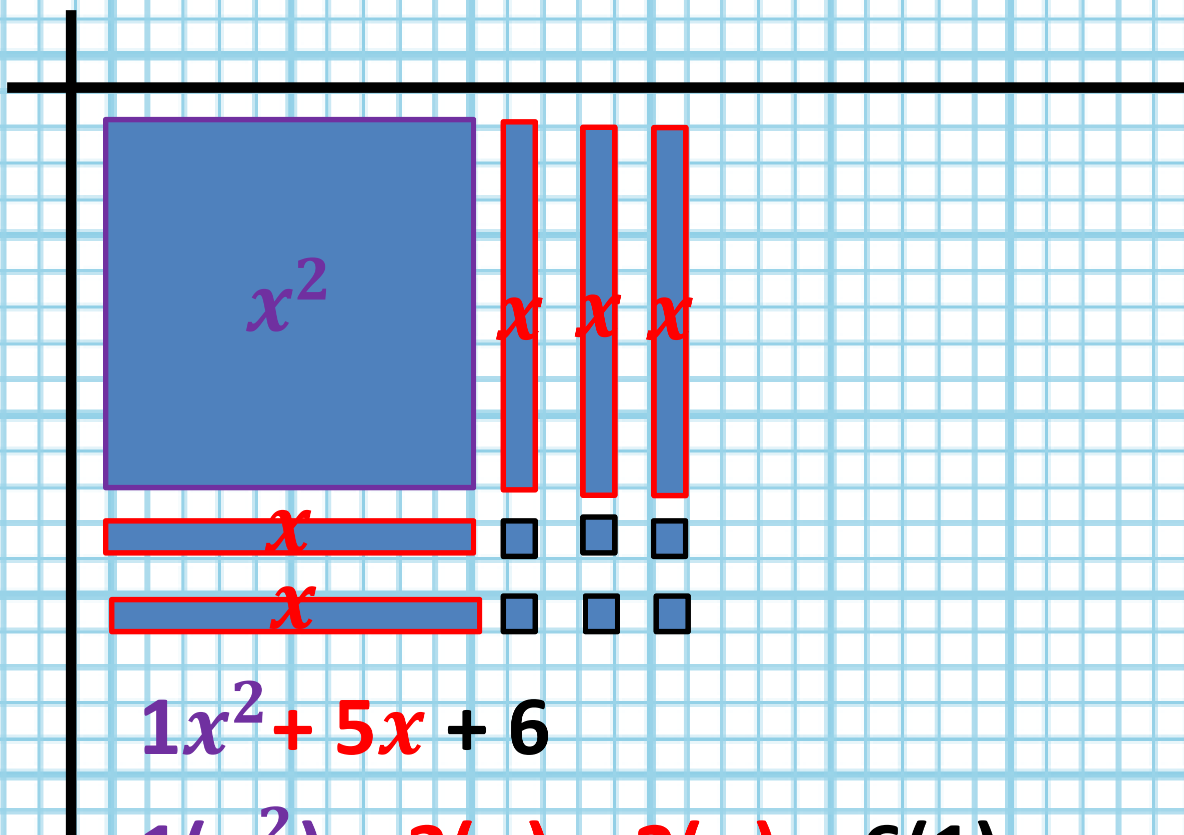
Try One: Factor $1x^2 + 5x + 6$ using Algebra Tiles



Draw the frame on your whiteboard leaving room to write in side lengths later and then factor $1x^2 + 5x + 6$.

Remember when you multiplied polynomials: Where were the quadratic terms? Where were the linear terms? Where were the constants?

Try One: Factor $1x^2 + 5x + 6$
Making the Rectangle

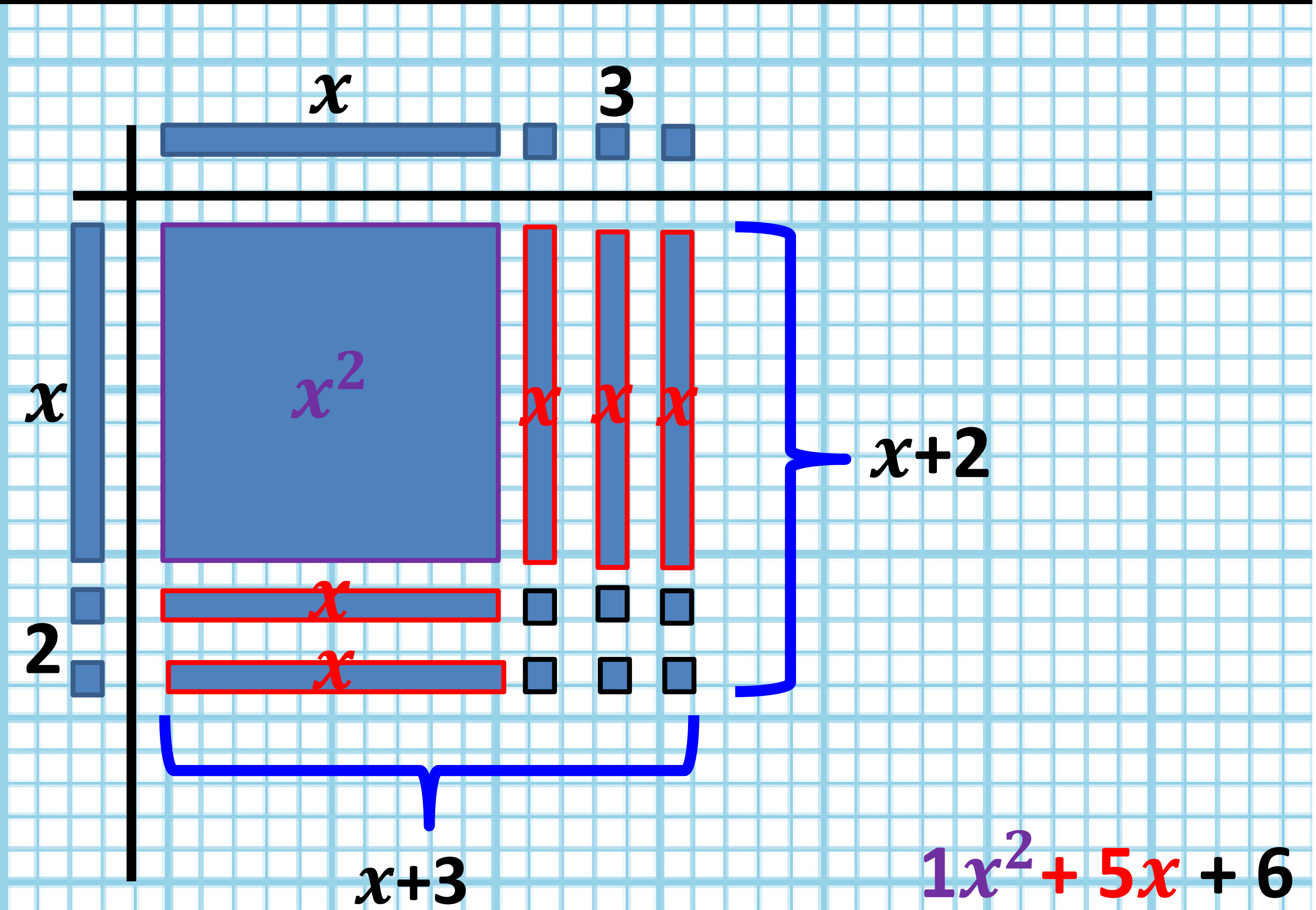


$1x^2 + 5x + 6$

$1(x^2) + 3(x) + 2(x) + 6(1)$

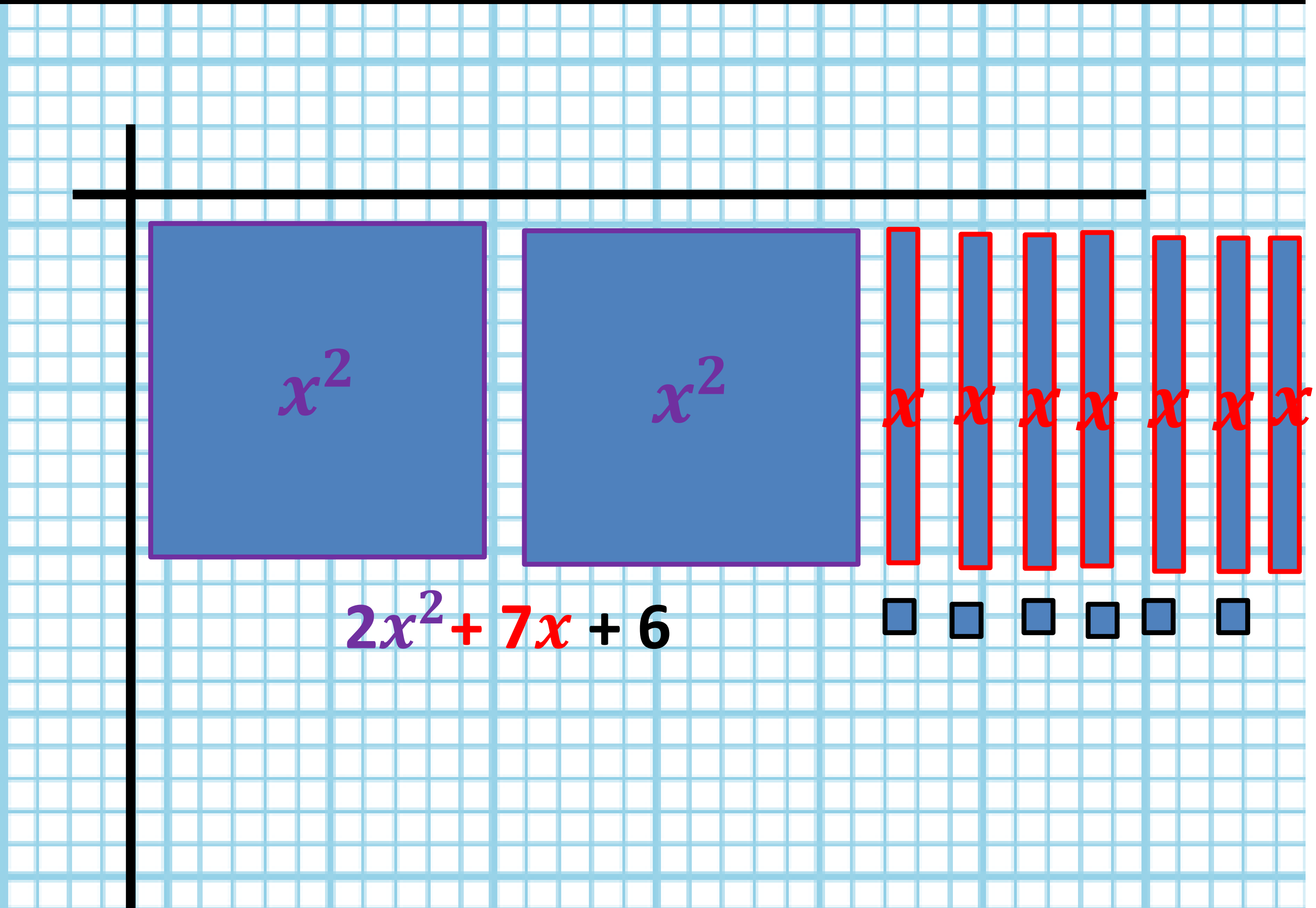
Remember the quadratics were on the top left and the constants are on the bottom right. Soon students realize the only possibilities for the sum of the linear terms are the factors of the constant, but which factors of the constant work? Read the lengths of each side of your rectangle and write in the factors on the frame on your whiteboard.

Try One: Factor $1x^2 + 5x + 6$
Reading the Side Lengths



Students often struggle to see the length of the sides on the top and on the left. If they cannot see it there, then use the opposite sides. Remind them it is the length of the side and not the area of the entire shape.

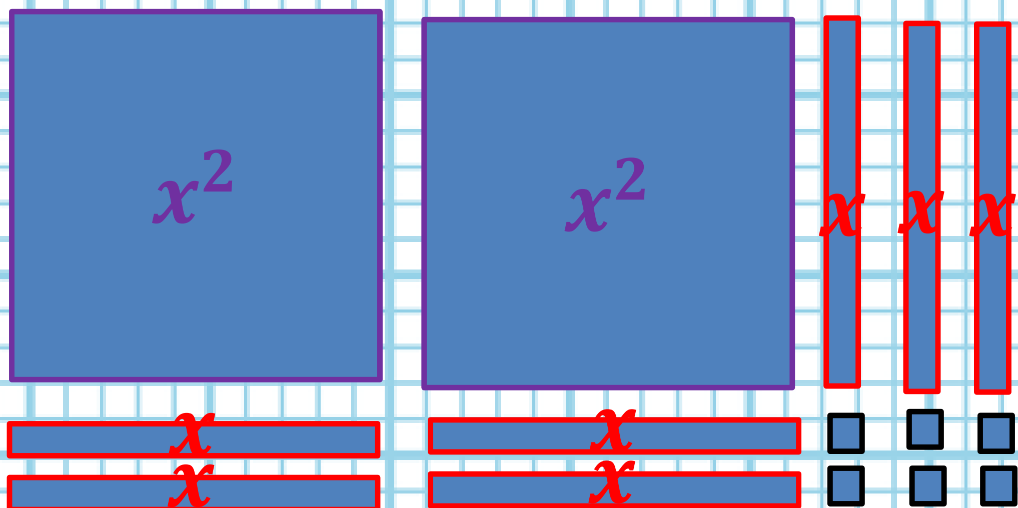
With a Partner: Factor $2x^2 + 7x + 6$ using Algebra Tiles



Remember multiplying polynomials and factor $2x^2 + 7x + 6$

(put it in a rectangle and find the side lengths)

With a Partner: Factor $2x^2 + 7x + 6$, The Rectangle



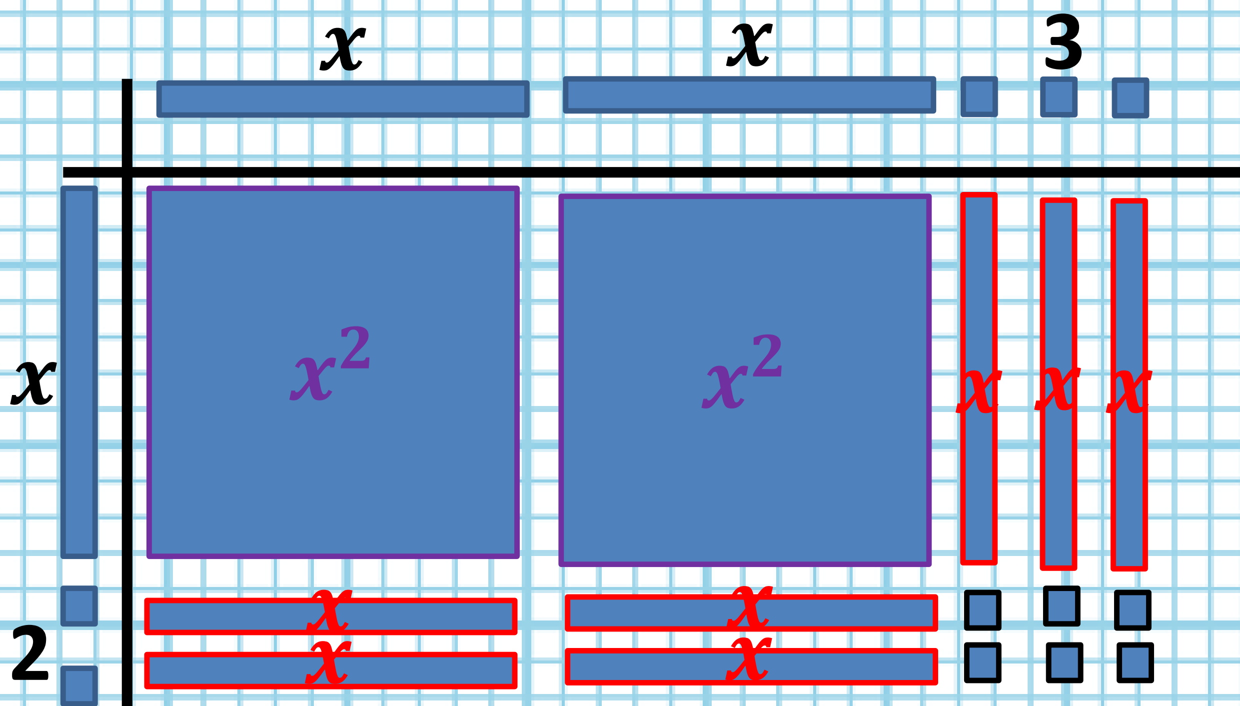
$2x^2 + 7x + 6$

$2(x^2) + 3(x) + 4(x) + 6(1)$

Emphasize that it has to multiply to make the quadratic and constant terms while adding correctly to make the linear terms. This is the X in X-box coming next.

Read the lengths of each side of your rectangle and write in the factors on the frame on your whiteboard.

With a Partner: Factor $2x^2 + 7x + 6$, The Rectangle



$2x^2 + 7x + 6$

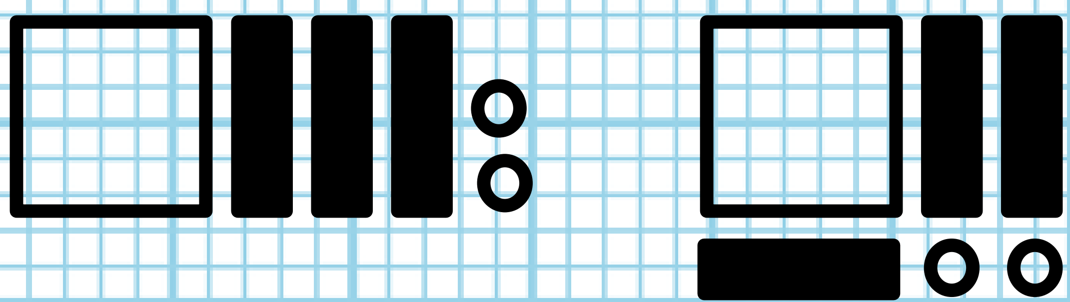
$2(x^2) + 3(x) + 4(x) + 6(1)$

$(2x + 3)(x + 2)$

The length of the two sides are your factors.

Emphasize that both expressions, $2x^2 + 7x + 6$ and $(2x + 3)(x + 2)$, are equivalent, both representing the area taken up by the shapes. By the way, less pain right?

Next Level of Abstraction: Draw the Shapes



Open terms are positive and closed terms are negative.

Referring to them as white being positive and black being negative is not a good idea.

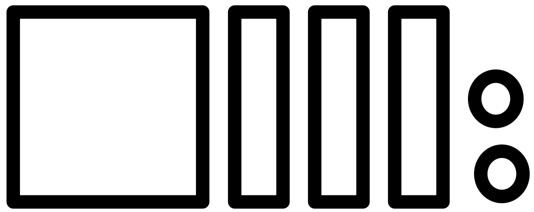
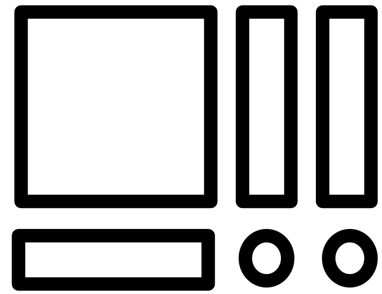
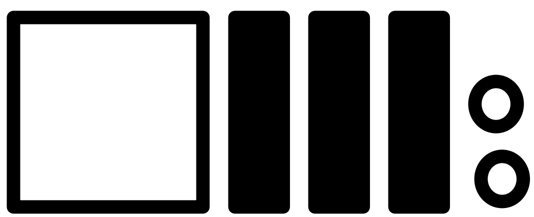
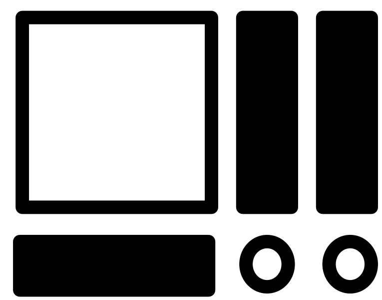
Fold Your Paper into Four Columns

Unfactored Expression	Pictorial Expression	Pictorial Factored	Factored Expression

Isn't this just the multiplication one backwards?

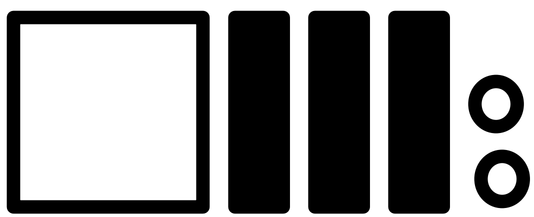
Exactly!

Factoring Polynomials: Four Column Notes

Unfactored Expression	Pictorial Expression	Pictorial Factored	Factored Expression
x^2+3x+2			$(x+2)(x+1)$
x^2-3x+2			$(x-2)(x-1)$

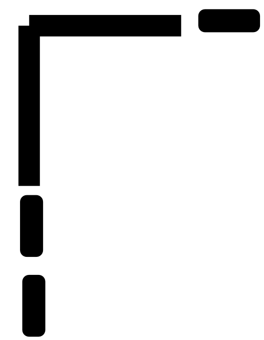
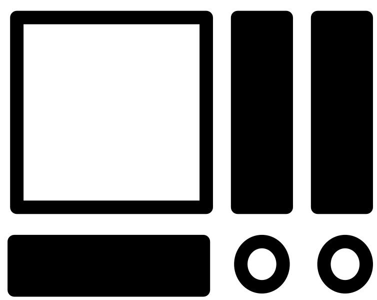
I included a negative term here as an example if you choose to go there before higher levels of abstraction.

Move Fluidly between Representations

Unfactored Expression	Pictorial Expression	Pictorial Factored	Factored Expression
x^2+3x+2			
			

Once they have practiced, you can give them a single expression and ask for the others.

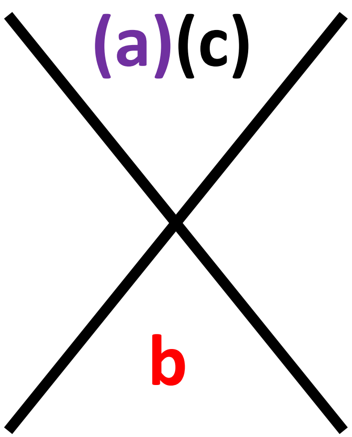
Move Fluidly between Representations

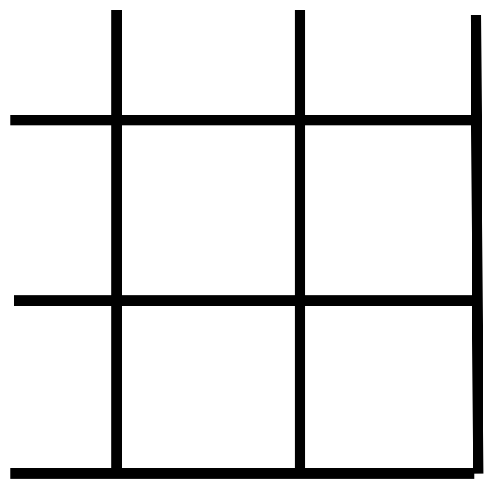
Unfactored Expression	Pictorial Expression	Pictorial Factored	Factored Expression
			$(x+2)(x+1)$
			

The top one is the most difficult so I included a method of drawing the length of each side before completing the picture.

Next Level of Abstraction: X Box

$ax^2 + bx + c$

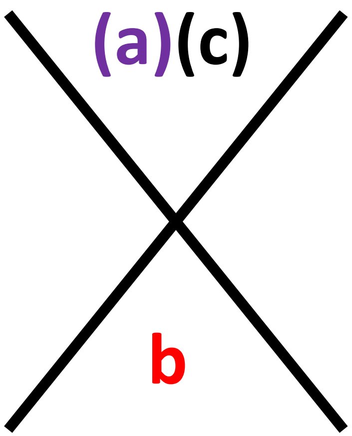




Sorry, no gaming today…

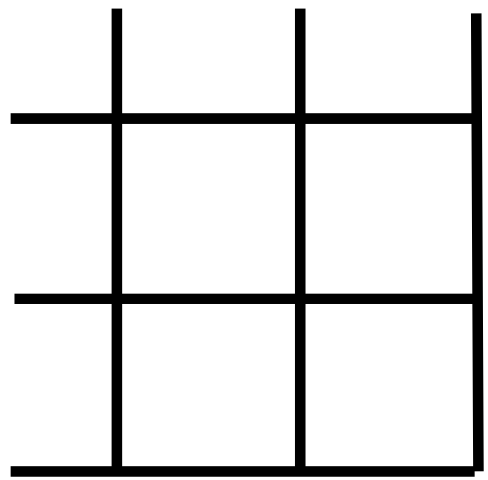
It is Not a Box; It is a Representation of Area

Linear Terms Must Multiply to make the Quadratic



Linear Terms Must Add (Combine) to Make the Final Linear Term

$ax^2 + bx + c$



You can use algebra tiles for negatives; however, in the interest of time now is good time to bring them in to the instruction.

The box method is a level of abstraction of the area method. Taught without the algebra tiles it means no more than numbers in a box like a multiplication table.

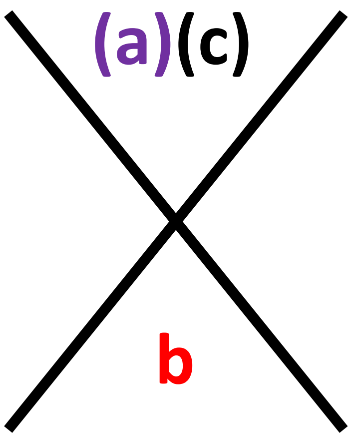
Algebra tiles should be done first so when students use the Xbox they visualize the algebra tiles.

Terms within the boxes represent the area of those boxes while the factors on the edge of the box represent the lengths of the corresponding sides.

Let's Play X-Box: Insert Quadratic & Constant

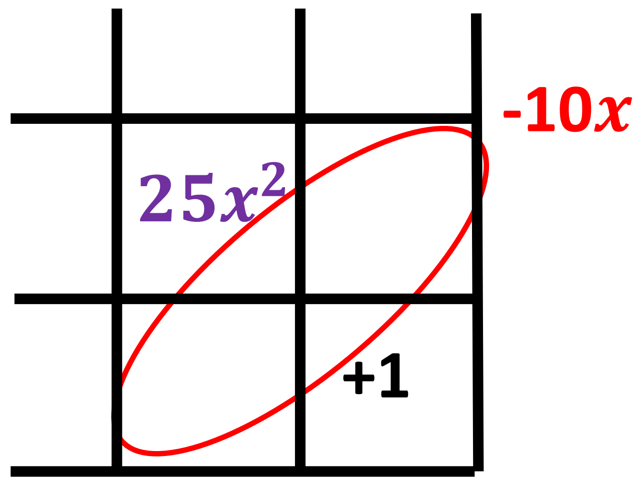
$ax^2 + bx + c$

Linear Terms Must Multiply to make the Quadratic



Linear Terms Must Add (Combine) to Make the Final Linear Term

$25x^2 - 10x + 1$



Now is also a good time to make the coefficients larger.

I extend the lines on the "box" to resemble length notation and make any negative signs more obvious, different than subtraction.

We are not really playing Xbox. We know the areas and are trying to find the factors, or the side lengths.

Remember the quadratic term goes in the top left and the constant goes in the bottom right. The question is how do I get -10x? There are an infinite number of ways to do so, which is why we have the X.

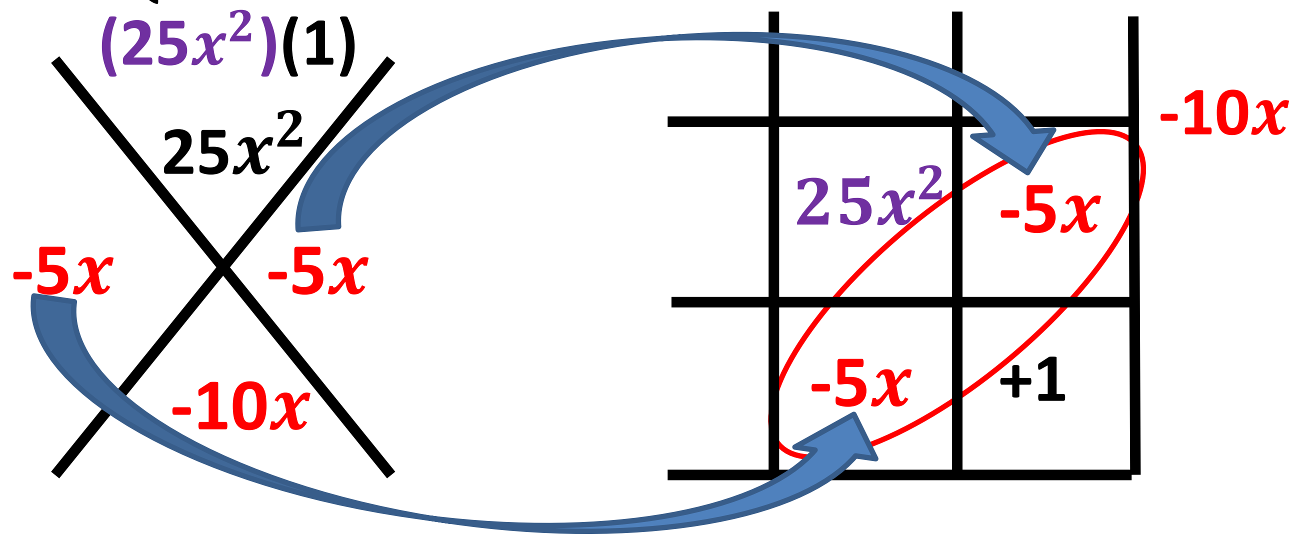
Let's Play X-Box: Use X to Find the Correct Linear Terms

$ax^2 + bx + c$

Linear Terms Must Multiply to make the Quadratic

$(25x^2)(1)$

$25x^2 - 10x + 1$



Linear Terms Must Add (Combine) to Make the Final Linear Term

These X's are awesome for practicing integer operations in seventh grade and preparing students for Algebra 1.

See X Marks the Spot by Brad Fulton at tttpress.com

What two numbers multiply to 25 and add (combine) to make -10?

You can use the coefficients only on the X as another level of abstraction.

By the way, this is a Perfect Square Trinomial. Difference of Two Squares, Perfect Square Trinomials, and Completing the Square should be done with algebra tiles first to allow students to discover the patterns or learn the rules. (SMP #8)

Let's Play X-Box: Find the Greatest Common Factor

$ax^2 + bx + c$

$25x^2 - 10x + 1$

Linear Terms Must Multiply to make the Quadratic

$(25x^2)(1)$

$25x^2$

$-5x$ $-5x$

$-10x$

Linear Terms Must Add (Combine) to Make the Final Linear Term

$5x$	$25x^2$	$-5x$
	$-5x$	$+1$

Be sure to factor out the GCF before doing the Xbox and the quadratic term should be positive.

If a=1, then you don't need the box; however, you want students to discover this to help them remember when to do it and when not to.

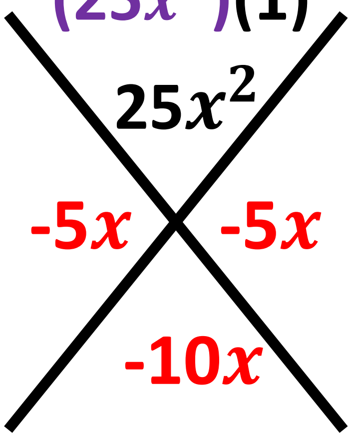
Find the Greatest Common Factor once for any row or column. Only take a negative if it is common to both .

Let's Play X-Box: Division or GCF Again

$ax^2 + bx + c$

Linear Terms Must Multiply to make the Quadratic

$(25x^2)(1)$



Linear Terms Must Add (Combine) to Make the Final Linear Term

$25x^2 - 10x + 1$

	$5x$	
$5x$	$25x^2$	$-5x$
	$-5x$	$+1$

Once you know a single length, then you can use it to find the rest or just keep using GCF.

Area is length times width so you can use this to find the rest of the side lengths.

What times $5x$ is $25x^2$?

Let's Play X-Box: Division or GCF Again

$ax^2 + bx + c$

Linear Terms Must Multiply to make the Quadratic

$(25x^2)(1)$

$25x^2$

$-5x$ $-5x$

$-10x$

Linear Terms Must Add (Combine) to Make the Final Linear Term

$25x^2 - 10x + 1$

	$5x$	-1
$5x$	$25x^2$	$-5x$
	$-5x$	$+1$

What times $5x$ is $-5x$?

Remember to point out you are using the area of the rectangle and one side to find the remaining side.

Let's Play X-Box: Division or GCF Again

$ax^2 + bx + c$

Linear Terms Must Multiply to make the Quadratic

$(25x^2)(1)$

$25x^2$

$-5x$ $-5x$

$-10x$

Linear Terms Must Add (Combine) to Make the Final Linear Term

$25x^2 - 10x + 1$

	$5x$	-1
$5x$	$25x^2$	$-5x$
-1	$-5x$	+1

What times $5x$ is $-5x$?

Of course, if students recognized this as a perfect square trinomial, then they would only have to do one side…

Let's Play X-Box: Check It !

$ax^2 + bx + c$

Linear Terms Must Multiply to make the Quadratic

$(25x^2)(1)$

$25x^2$

$-5x$ $-5x$

$-10x$

Linear Terms Must Add (Combine) to Make the Final Linear Term

$25x^2 - 10x + 1$

	$5x$	-1
$5x$	$25x^2$	$-5x$
-1	$-5x$	$+1$

If it has been done correctly, then the final area should check out.

Does -1 times -1 equal +1?

By the way, the base ten equivalent, if x=10, would be

(49)(49)

(50-1)(50-1)

2500-50-50+1

2401

… a great way to introduce special products!

Final Level of Abstraction: Just do it! (Guess and Check)

$25x^2 - 10x + 1$

$(5x - 1)(5x - 1)$

or

$(5x - 1)^2$

How much can guess and check be informed by using algebra tiles?

1 is prime so it has to be…

(1)(1)

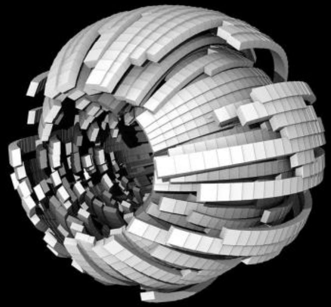
I need the linear terms in the product to be negative so it has to be…

(-1)(-1)

25 factors into (25)(1) or (5)(5); (25)(1) would add to 26 so it must be…

$(5x - 1)(5x - 1)$ or $(5x - 1)^2$

Special Cases: Using Algebra to Develop Number Sense



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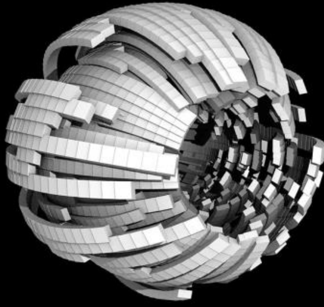
Meaning for Memory



Mentally, find the product:

$$49 \cdot 49$$

How can we do this without paper?

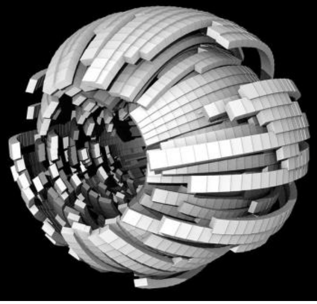
Did you get an answer?

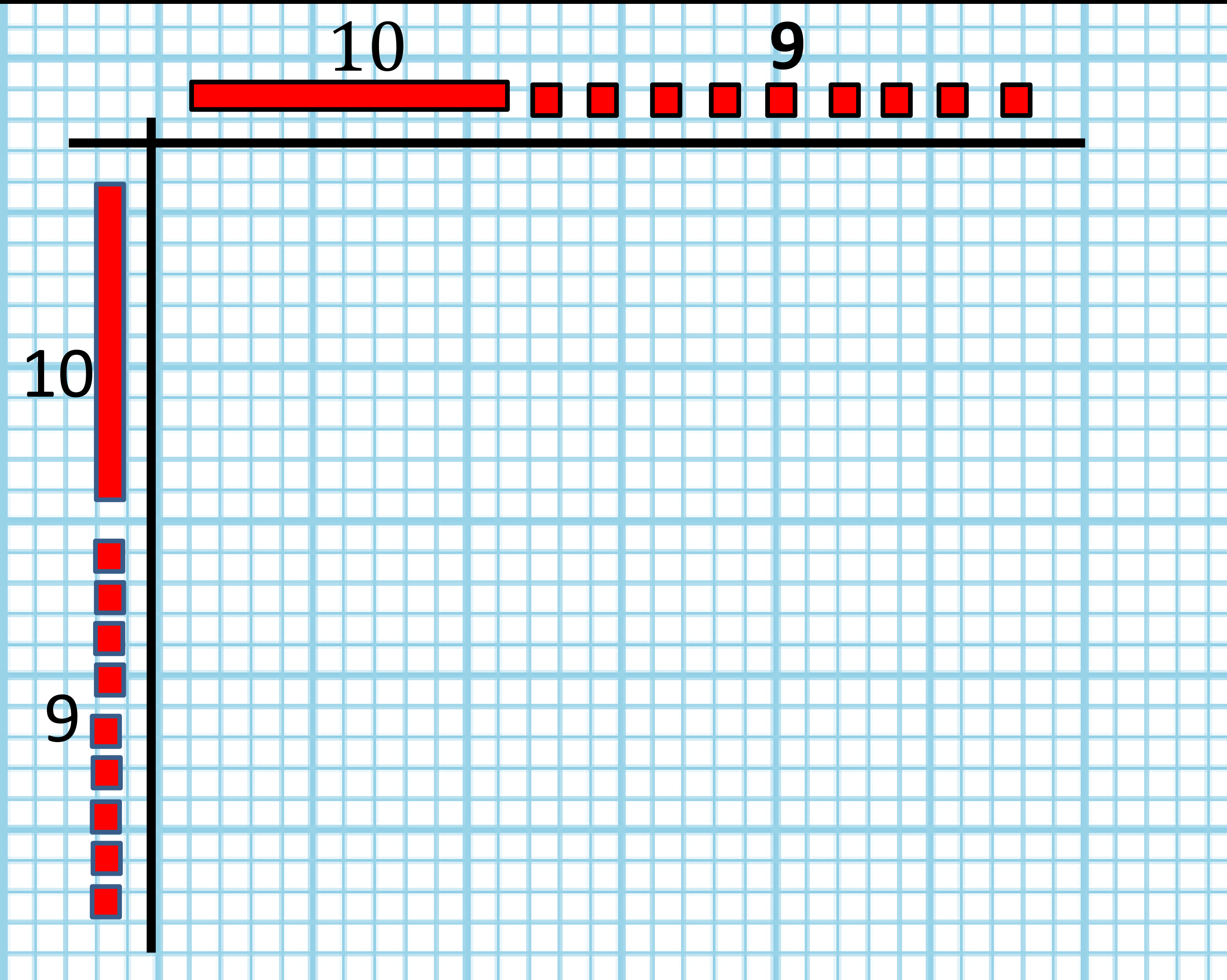
Did anyone skywrite?

Maybe you used an area method, FOIL, or the distributive property?

Is there another way?

Try One: Find the Product of 19x19 = (10 + 9)(10 + 9) Using Base Ten Tiles

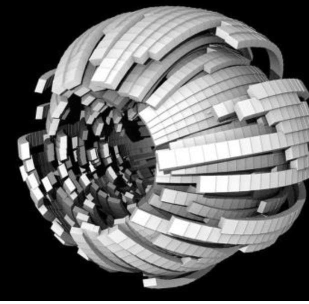


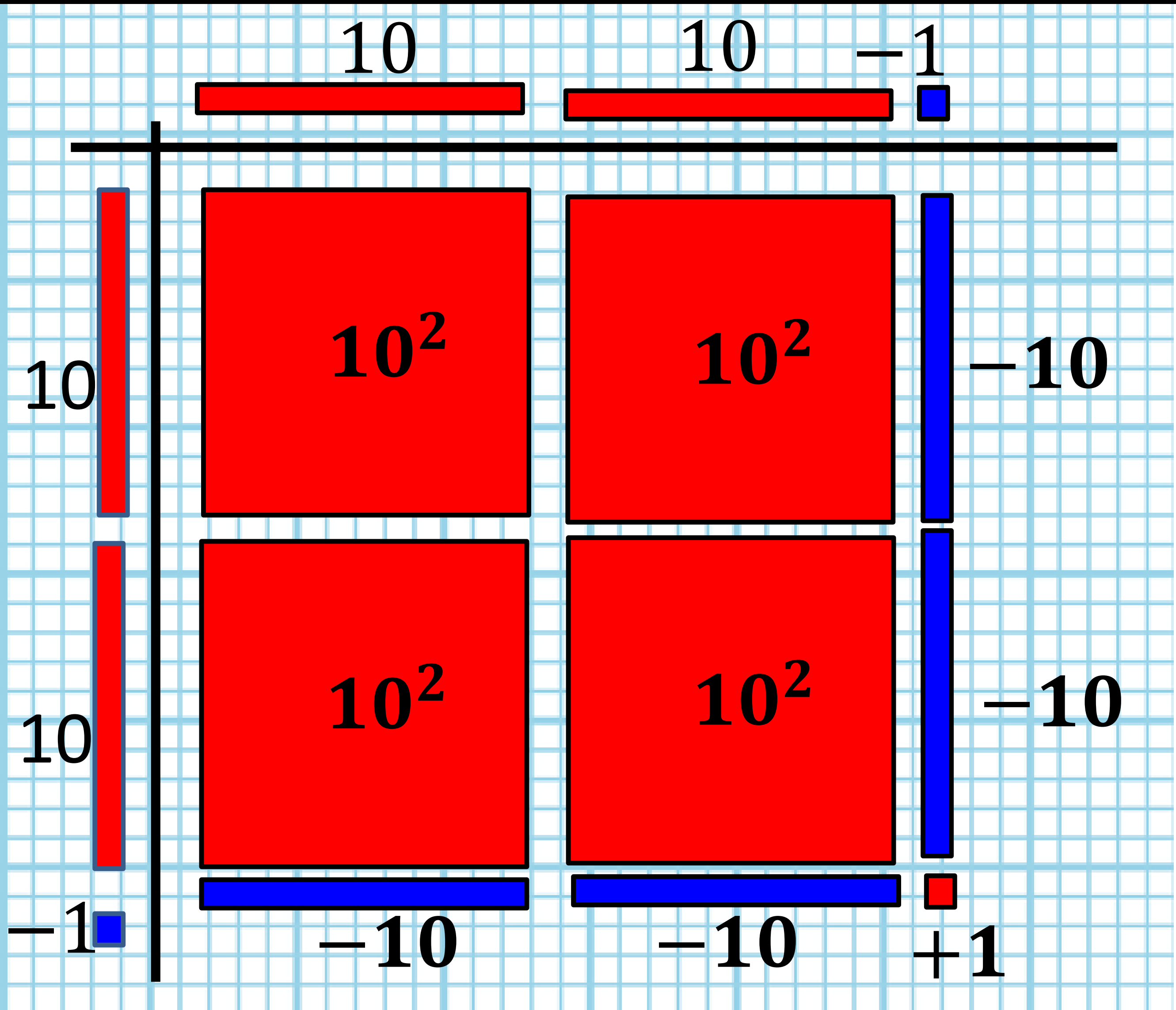


This looks like a lot of work…

Is there another way?

Try One: Find the Product of 19x19 = (20 – 1)(20 -1) Using Base Ten Tiles





$20^2 - 1(20)$
$- 1(20) + 1$

$20^2 - 2(20)$
$+ (-1)^2$

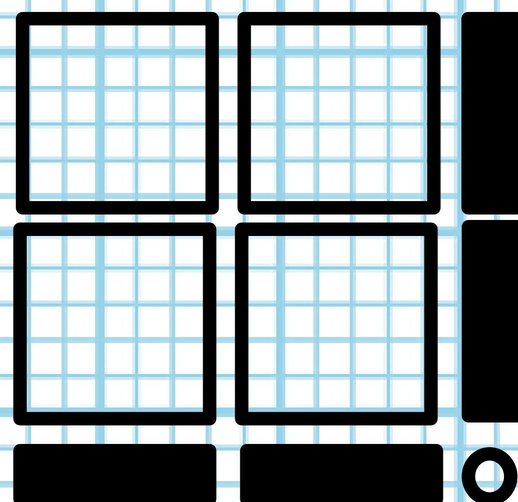
400-20-20+1

400-40+1

360+1

361

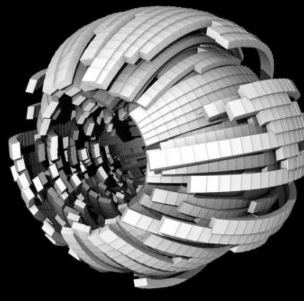
Next Level of Abstraction: Draw the Shapes

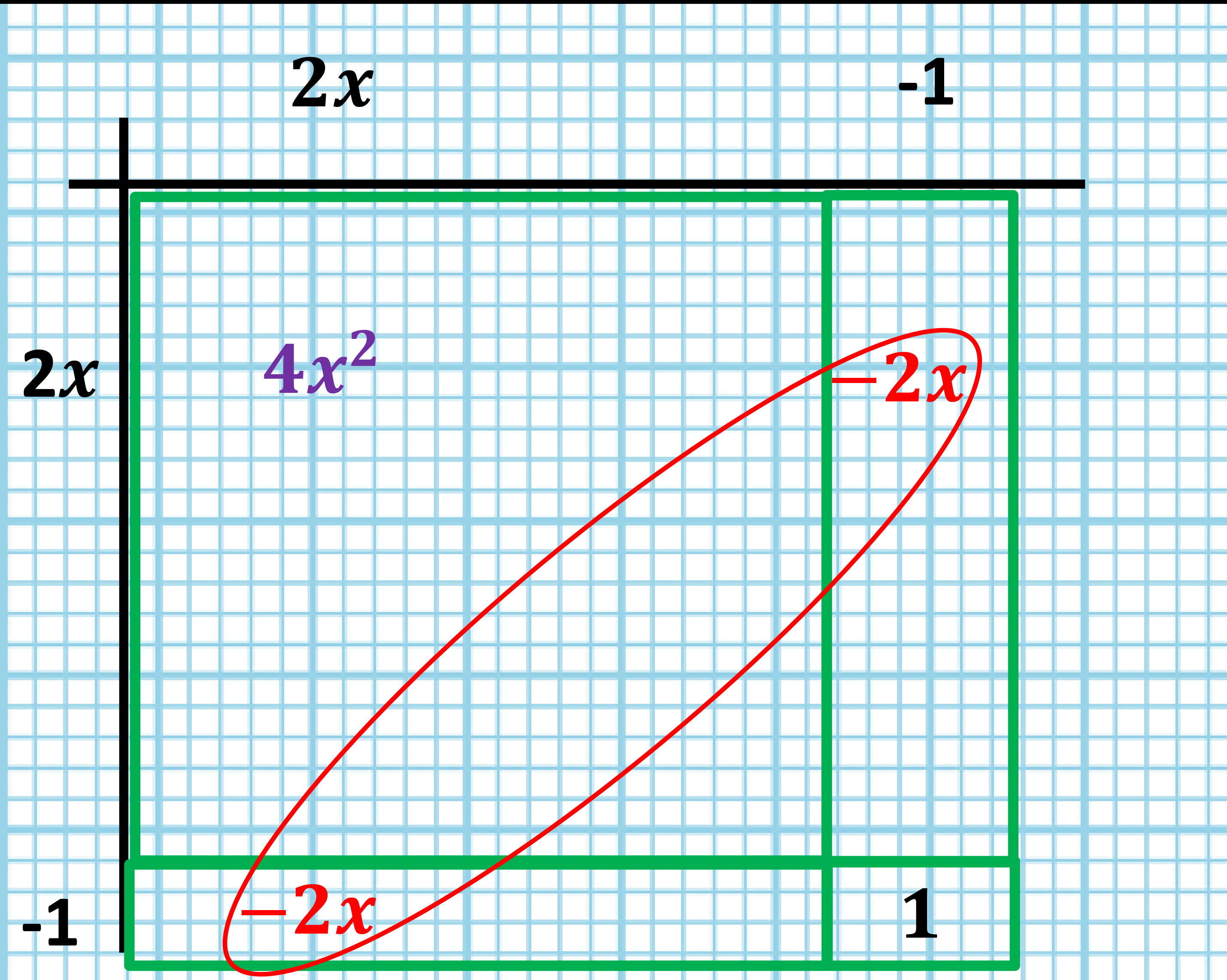


Open terms are positive and closed terms are negative.

Referring to them as white being positive and black being negative is not a good idea.

Next Level of Abstraction Area Representation without the Tiles (to scale)





I rarely use this abstraction step; however, I might show it once to students.

Box Method?

The terms represent the tiles or the area of the rectangle they are within, which needs to be reinforced. The are not merely entries in a table or box.

Still $(2x - 1)^2$

$(2x - 1)(2x - 1)$

$4(x^2)$ - $2(x)$ - $2(x)$ + (1)

$4x^2$- $4x$ + 6

Emphasize combining like terms on the diagonal

Next Level of Abstraction Area Representation without the Tiles (not to scale)

	$2x$	-1
$2x$	$4x^2$	$-2x$
-1	$-2x$	$+1$

$$(2x-1)^2$$
$$(2x-1)(2x-1)$$
$$4x^2-2x-2x+1$$
$$4x^2-4x+1$$

By the way, negatives are effective for base 10 and mental multiplication like (50-1)(50-1) instead of (49)(49).

I like to extend the lines outside of the large, exterior rectangle to help students see negative signs, and it makes them resemble the tick marks indicating length.

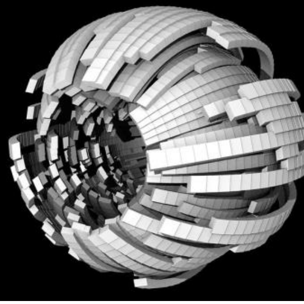
Why is 2 not the best coefficient to use if your goal is to find a pattern?

Spoiler alert:

(2)(2) = 4 and

2+2 = 4

Final Level of Abstraction
The Distributive Property



	$2x$	-1
$2x$	$4x^2$	$-2x$
-1	$-2x$	$+1$

$$(2x-1)^2$$

$4x^2$ $+1$

$$(2x-1)(2x-1)$$

$-2x$

$-2x$

$$4x^2 - 2x - 2x + 1$$

$$4x^2 - 4x + 1$$

Students should be able visualize or use their experience with the area representation and/or the box to distribute correctly.

Fewer students forget to smile this way. In other words, fewer students do FL instead of FOIL.

Don't forget to smile!

Work the diagonal.

Find and combine your linear terms

Generalizing to the Formula

	ax	$-b$
ax	a^2x^2	$-abx$
$-b$	$-abx$	$+b^2$

$$(ax-b)^2$$

$$(ax-b)(ax-b)$$

$$(ax)^2-1abx-1abx+b^2$$

$$(ax)^2-2abx+b^2$$

In a way, this is actually the final level of abstraction. All numbers are replaced by variables, which is a major abstraction.

Mentally, find the product:

$$49 \cdot 49$$

What Materials Did I Use?

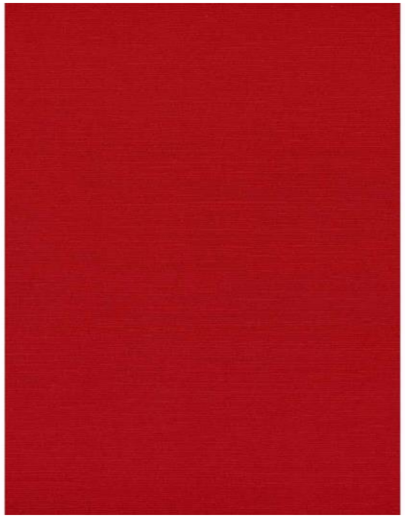
Two colors of cardstock

Slider bags

Folded blank paper

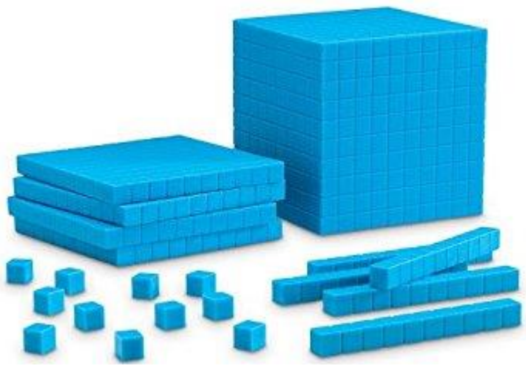
Cheapest manipulatives EVER!

And they do much more than this…



What about the Base Ten Blocks?

Some might argue against having the same manipulative as base ten and algebra tiles.



My counter argument is why keep two manipulatives when one will do?

More importantly, I *want* them to make connections between base ten and algebraic representations of any base to aid in understanding and retention.

Color and Algebra Tiles?

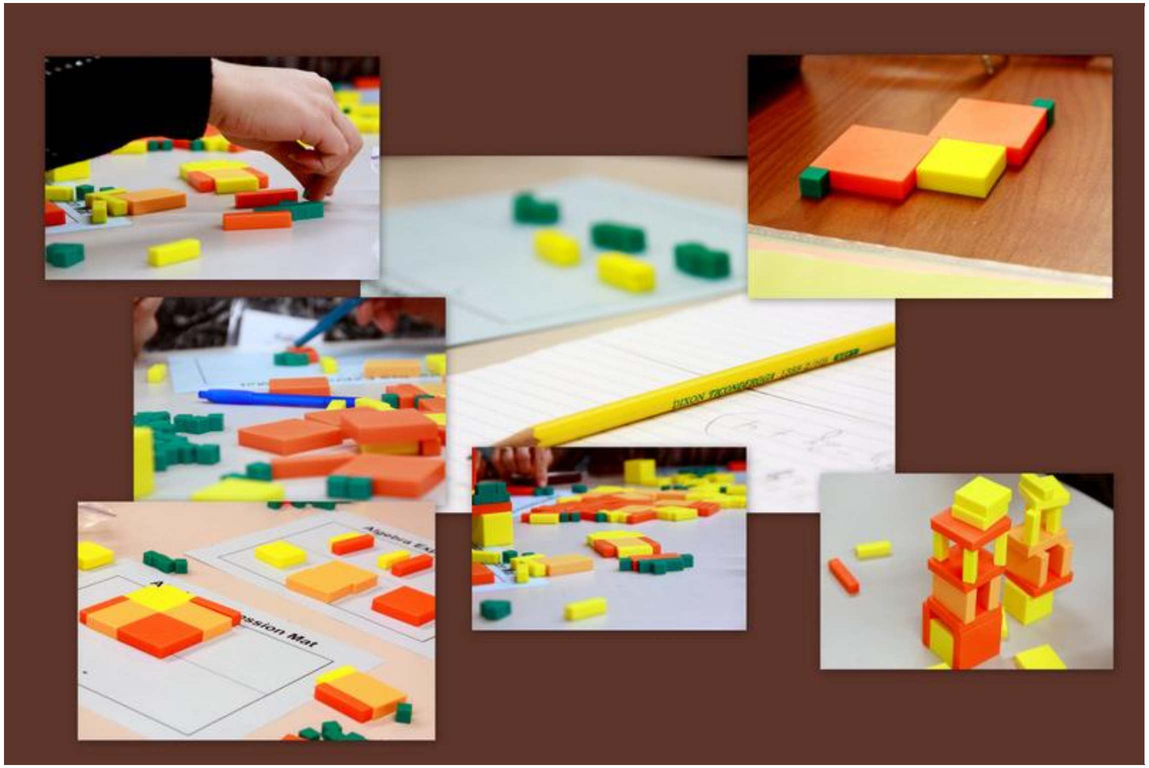
Store bought algebra tiles have "unknown" side lengths as they are trying to reinforce the that quality of the variable.

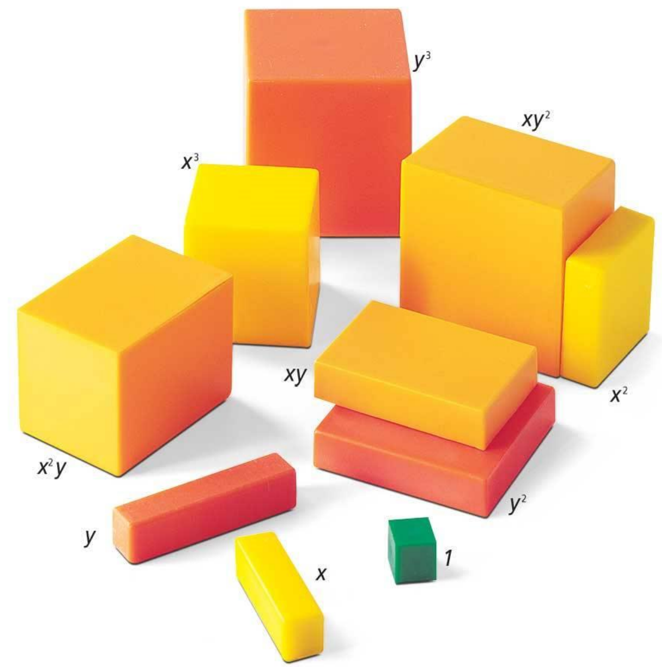
If you measure these tiles with the one, then you find they are usually between three and four. Does that mean it has a fractional side length?

All models have inherent flaws; choose the flaws that reinforce the learning when you can.

I don't prefer the multiple color tiles. I prefer to use color to distinguish between positive and negative or different variables.

What about Y? Color & Algeblocks





AlgeBlocks allow for work in three dimensions, cubic expressions & equations, and work with two variables, x & y.

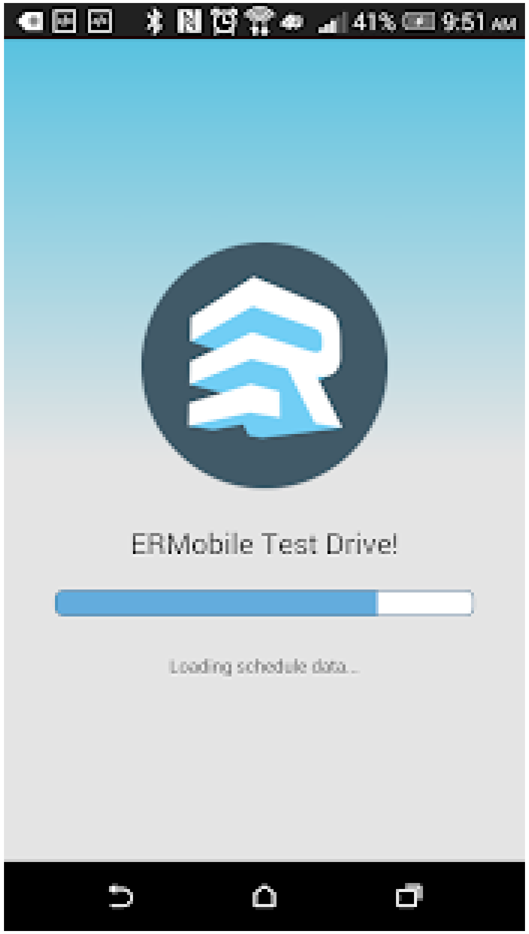
http://www.hand2mind.com/item/algeblocks-manipulative-starter-set/9241 Expensive @ $200 though.

Presentation, Questions & Feedback

Please complete the session survey via the ERMobile Conference App.

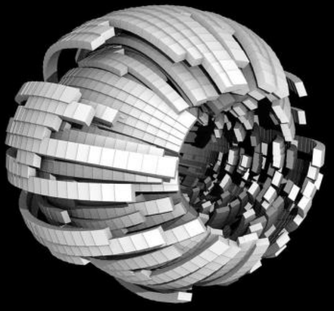
This presentation along with others are posted on my fledgling website www.meaningformemory.com

Please leave other feedback & questions on the site or email me at david.mattoon@gmail.com



Handouts

David Mattoon
Meaning for Memory



What are our words worth?

"I'll give you a thousand words for it."

A Picture is Worth a Thousand Words ~ Fred Barnard, 1927

The notion that a **complex idea** can be conveyed with just **a single still image** or that **an image of a subject conveys its meaning** or essence **more effectively** than a description does.

Ever get tired of repeating yourself ?

- "**Hearing something a hundred times** isn't better than seeing it once"

  ~ Chinese Expression

- **“The drawing shows me at a glance what would be spread over ten pages in a book.”** ~ Ivan Turgenev, Russian Author
- **"A good sketch is better than a long speech“** ~ Napoleon Bonaparte, Emperor of the French