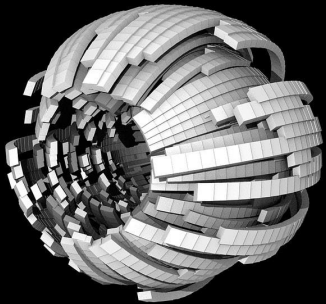
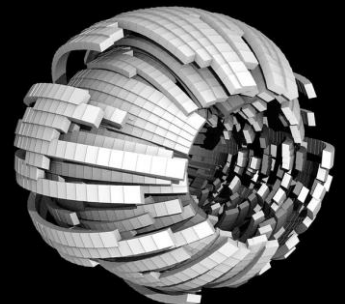


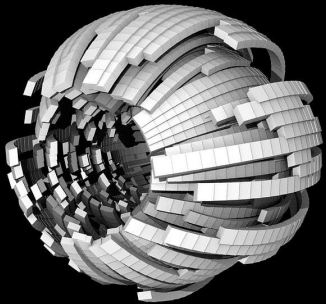
Pictorial Mathematics: Polynomial Multiplication & Factoring



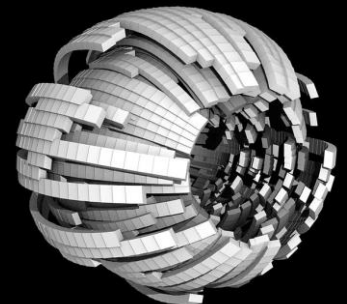
David Mattoon
Meaning for Memory
Fall for Math 2018



Why Concreteness Fading Time Rigor



David Mattoon
Meaning for Memory



Deficiency: Students or Environment?

In educational settings, this construction of dis/ability manifests in the double education system that splits general education and special education. Scholars have traced the ways in which special education **“serves as a vehicle for preserving general education in the midst of ever increasing diversity”** (Reid & Valle, 2004, p. 468, paraphrasing Dudley-Marling, 2001; also see Skrtic, 1991, 2005). **Rather than using research-validated frameworks** like Universal Design for Learning (UDL) and Complex Instruction (CI) **to deliver rigorous, high-cognitive demand instruction to all mathematics students**, the system of special education **shunts certain students (especially students of color) into an inferior, segregated mathematics education, thus providing a band-aid to a broken general education system and preventing larger, more systematic changes.**

Returning to the assumptions inherent in the concept of intervention, a disability studies perspective problematizes **the taken-for-granted assumption that what is “wrong” with the situation requiring intervention is a pathology or deficit within students. Instead, the problem is located in the inaccessibility of the environment; in other words, what needs to be changed is not the student, but rather the environment to allow access for students who differ from one another.** As Reid and Valle (2004) assert, “the responsibility for ‘fitting in’ has more to do with changing public attitudes and the development of welcoming classroom communities and with compensatory and differentiated instructional approaches than with individual learners (Shapiro, 1999). In other words, **our focus is on redesigning the context, not on ‘curing’ or ‘remediating’ individuals’ impairments**” (p. 468). A related line of research of working group members involves conceptualizing interventions into participation rather than content. That is, **what interventions might contribute to more equitable participation and deeper engagement across students in mathematics classrooms?** For example, one of the working group members has conducted empirical research focused on equitable participation in a Cognitively Guided Instruction algebra routine. Moreover, a political/relational model suggests that inaccessibility is embedded in the context of power relations. **Finding ways to “intervene” to make the environment accessible, then, also requires analyzing the power relations involved** in maintaining inaccessibility.

<http://jamessheldon.com/reframing-interventions-in-mathematics-education-emerging-critical-perspectives/>

Change the Environment



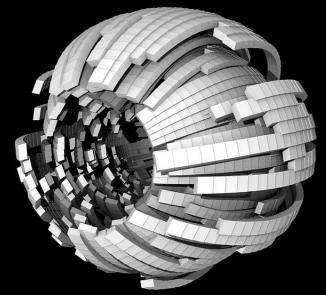
David Mattoon

Meaning for Memory



Levels of Abstraction = Concreteness Fading

Concrete, Representational, Abstract (CRA)



During the first half of my teaching career, I would spend what seemed to be the first half of a math lesson teaching a new math concept by sharing definitions, formulas, steps and procedures.

To make things more challenging for my students, I would simultaneously introduce the symbolic notation used to represent those ideas. Then, I would spend the **remainder** of the lesson **attempting** to help my students make sense of these very **new** and often **abstract ideas**.

By the end of the lesson, I could help many students build an understanding, but there was **always a group I felt who I would leave behind**. Like many other teachers, I was just **teaching in a very similar way to that how I was taught. I knew no different.**

However, if we consider that new learning requires the linking of new information with information they already know and understand, we should be intentionally planning our lessons with this in mind. **A great place to start new learning is through the use of a meaningful context and utilizing concrete manipulatives that students can touch and feel.**

When we teach in this way, we **minimize the level of abstraction** so students can **focus their working memory on the new idea** being introduced in a **meaningful way**.

- Kyle Pierce, *Tap into Teen Minds*, <https://tapintoteenminds.com/concreteness-fading/>

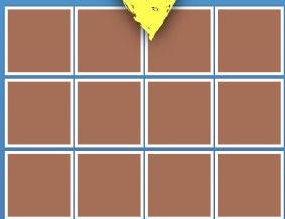
Concreteness Fading

How many donuts are in 4 boxes of 12 donuts?

1 Concrete

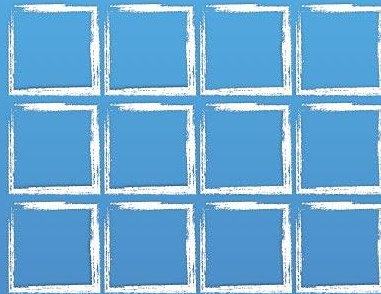


Actual
Doughnuts



Concrete
Manipulatives

2 Visual



Drawings and
Diagrams

3 Abstract

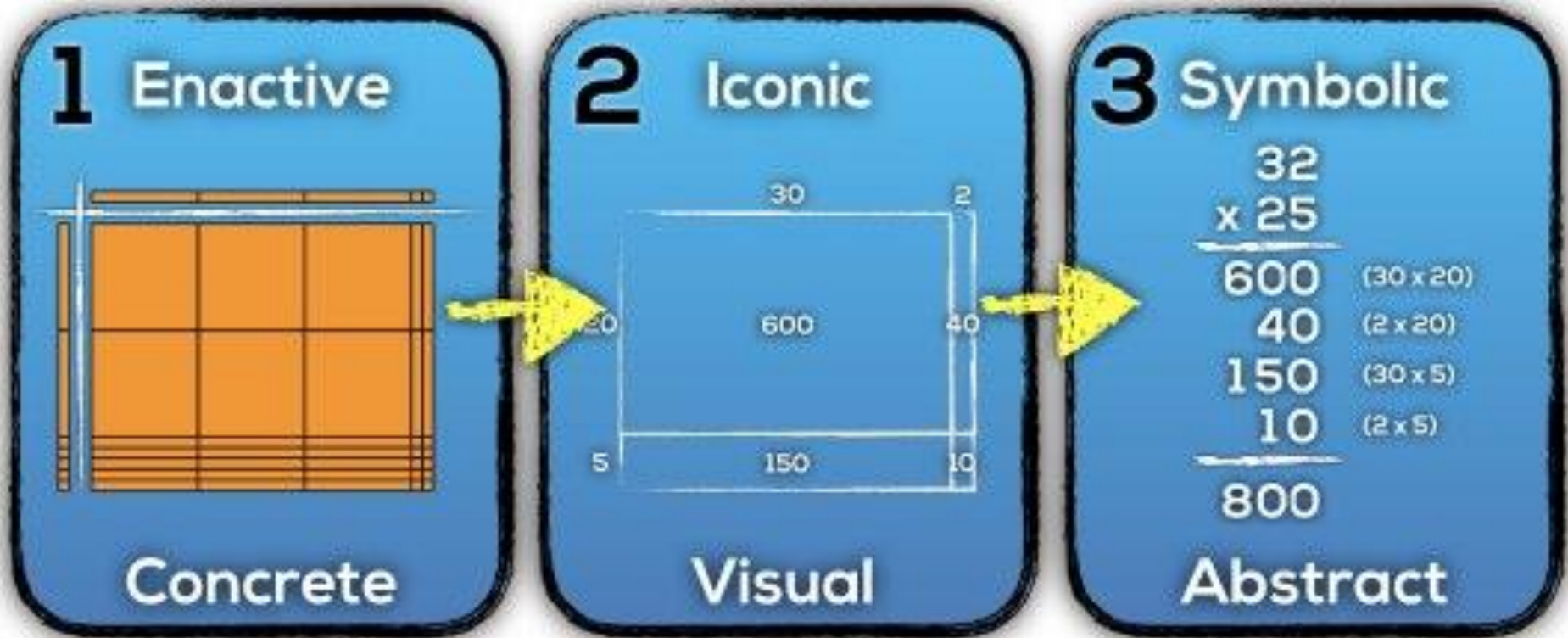
3 groups of
4 doughnuts
is equal to
12 doughnuts

Symbolic

$$3 \times 4 = 12$$

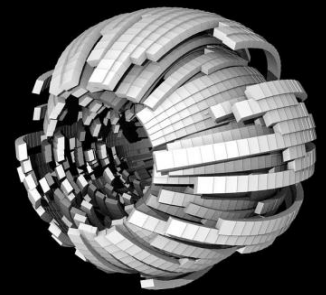
Concreteness Fading

How many doughnuts are in the giant box?

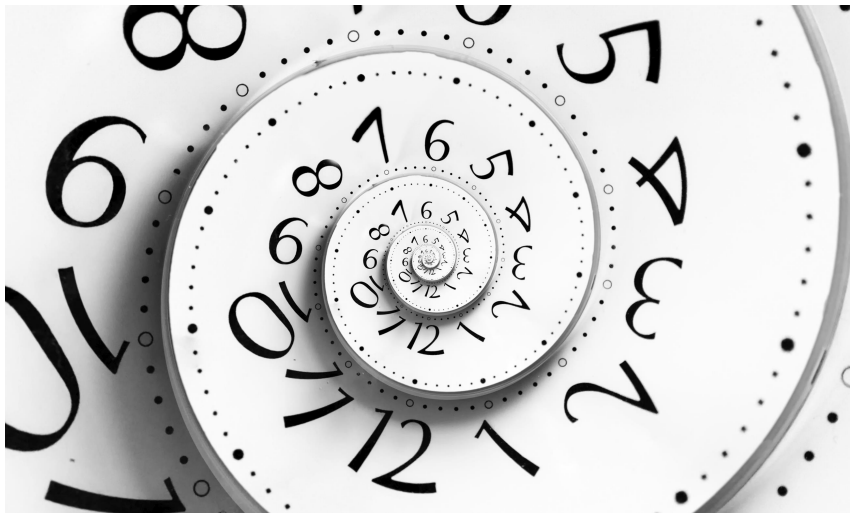
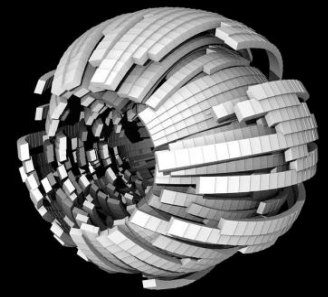


Jerome Bruner (1966) proposed three modes of representation:
Enactive representation (action-based)
Iconic representation (image-based)
Symbolic representation (language-based)

Time: I don't have it.



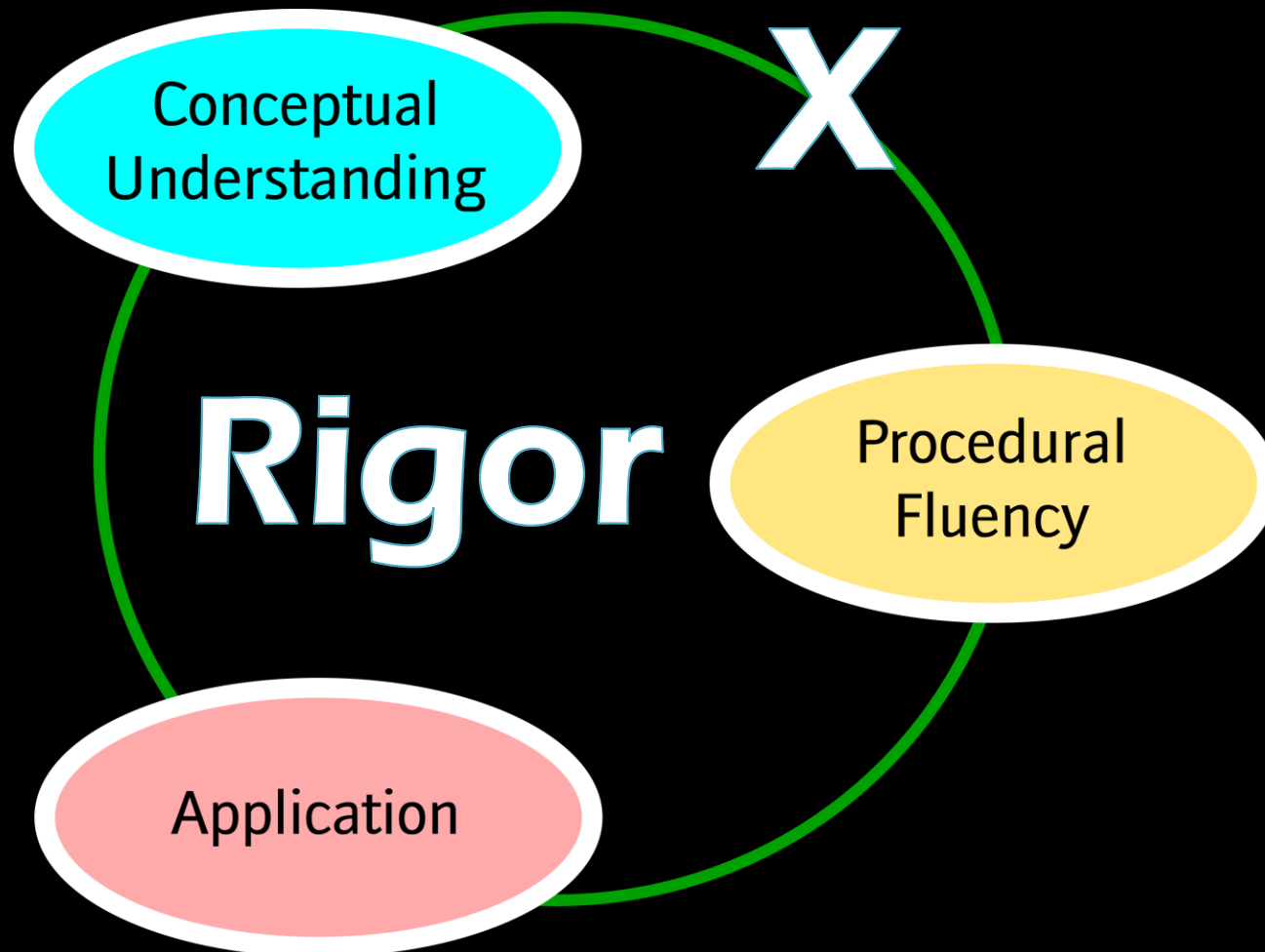
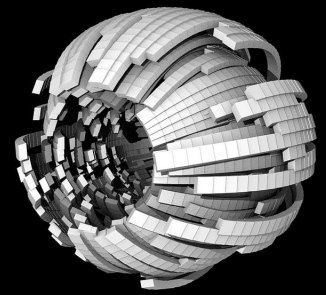
I might make time to...



Pictorial Math will save you time in the long run as it will aid students in understanding what to do, why they do it, and how to transfer the knowledge.

- Review in the beginning of the year
- Introduce new material by re-teaching material from former grade levels
 - I am not talking about making connections to former material here.
- Reteach for full periods when they don't understand
- Reteach for full periods when they cannot remember
 - I am not talking about interleaving
- Teach multiple procedures for slightly different problems
 - I am not saying procedures are bad.

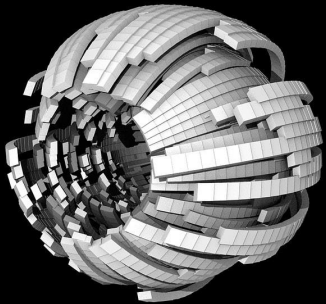
Where does this presentation live?



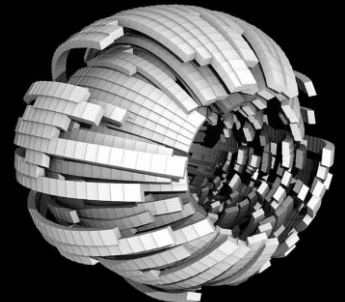
David Mattoon

Meaning for Memory

Polynomial Combination: A Necessary First Step



David Mattoon
Meaning for Memory



Algebra Tiles

1



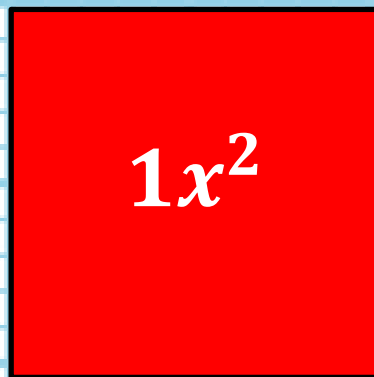
$1x$



*Constant –
a length of
1 and a
height of 1*

*Linear –
a line of
length x
and a
height of 1*

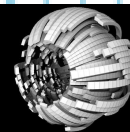
$1x^2$



*Quadratic –
a square
with a side
length of x .*

In the case of algebra tiles, one is the small square, which is the same as base ten. You might say it is constantly one. Really it is a constant, because the value of x will not influence it.

Begin referring to x squared as a square with a side length of x . A square with two dimensions, a length of x and a width of x .



Algebra Tiles

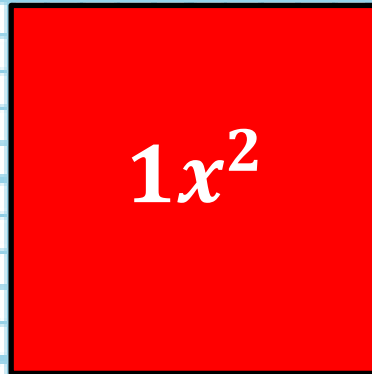
$1x^0$



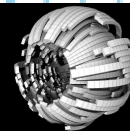
$1x^1$



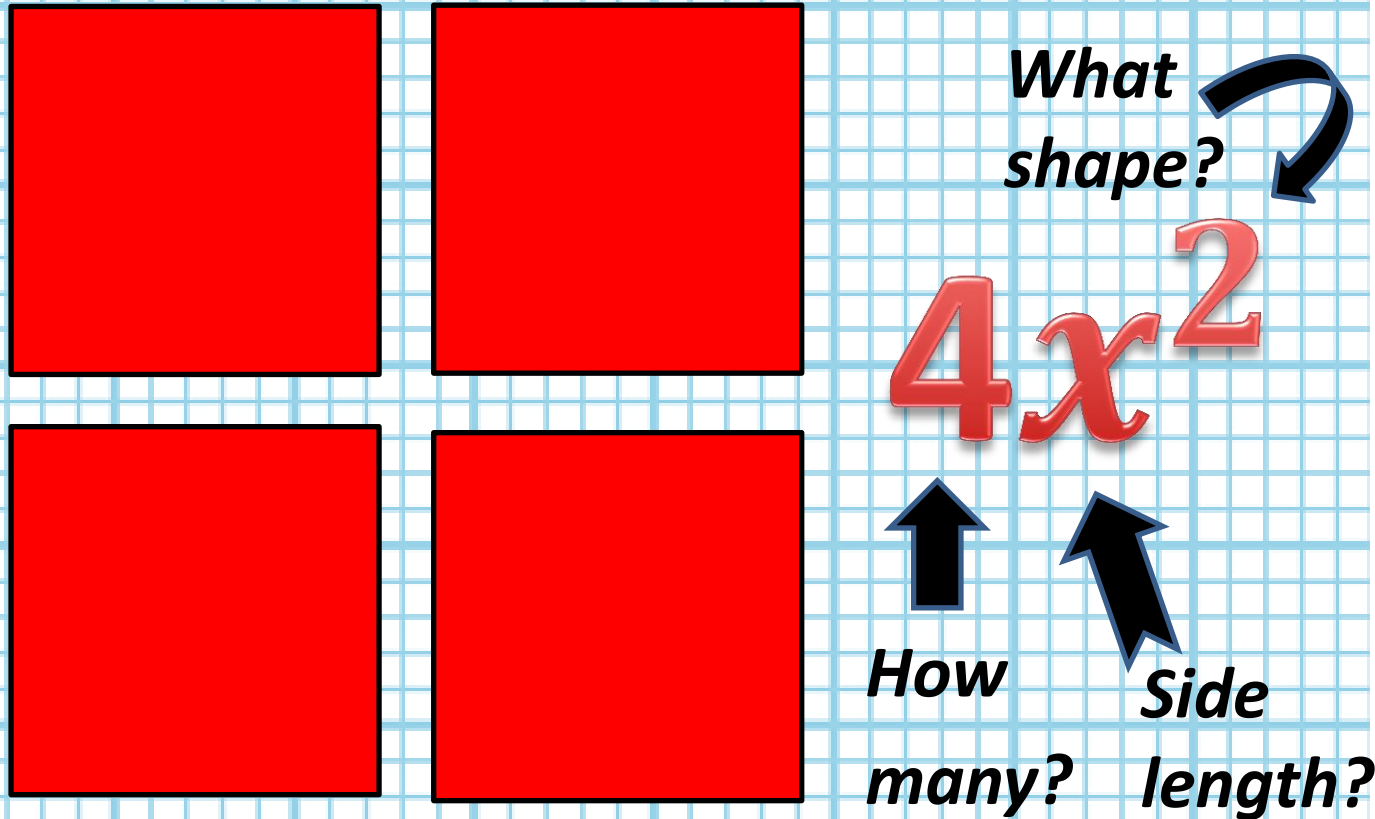
$1x^2$



Our number system is based on powers of ten; therefore, it is the exponent, which drives what shape it is.



Anatomy of an Algebraic Term

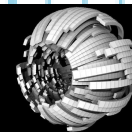


You want students to discover this and/or be able to tell you when you ask:

What does the exponent mean?

What does the base mean?

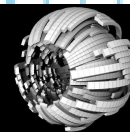
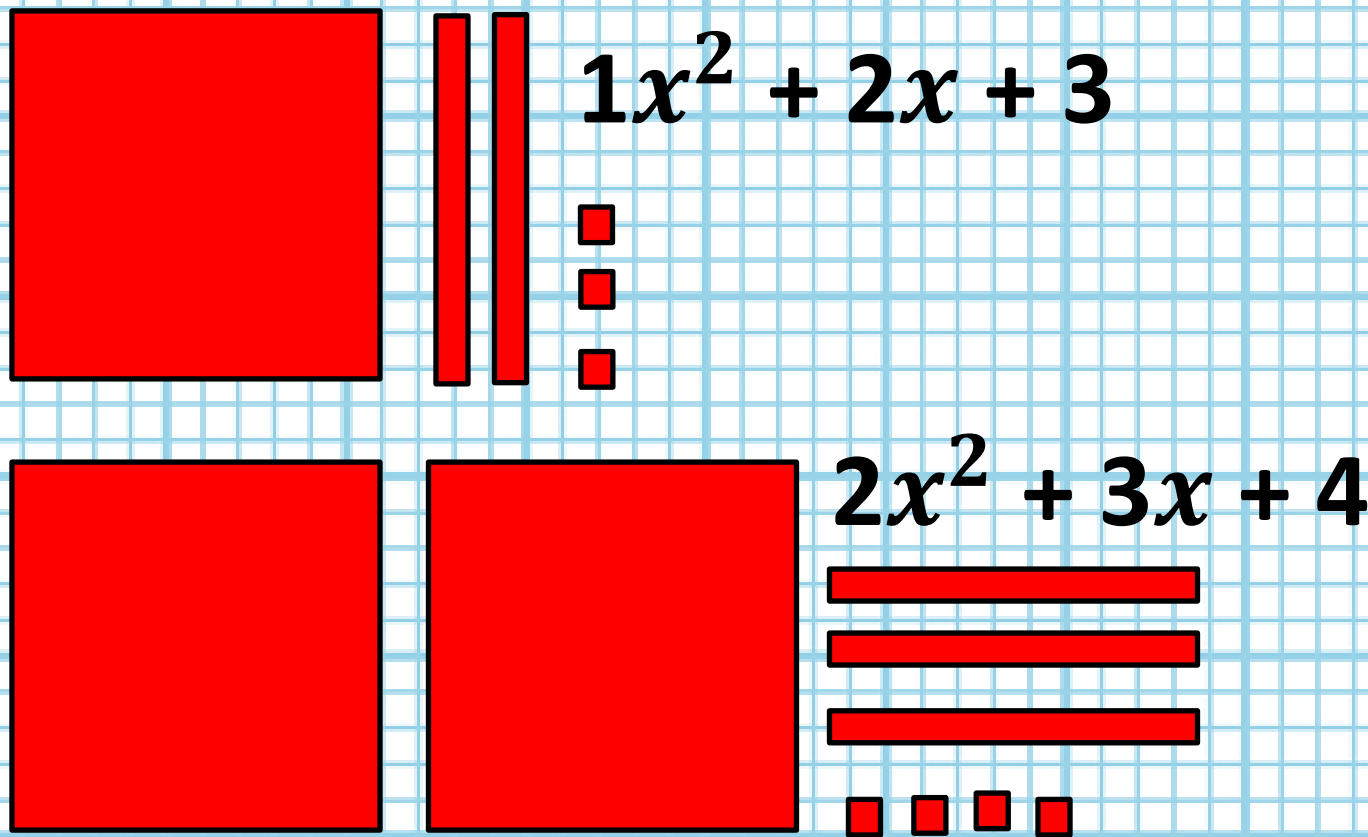
What does the coefficient mean?



Algebra Tiles: Addition

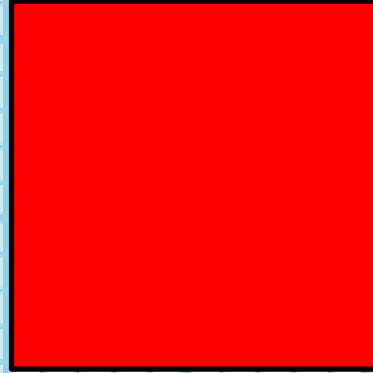
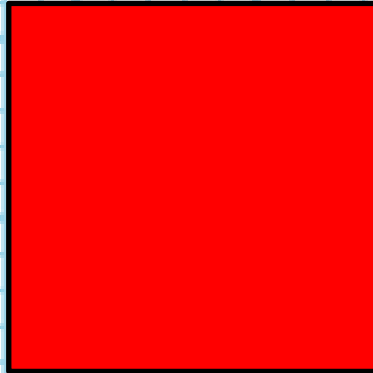
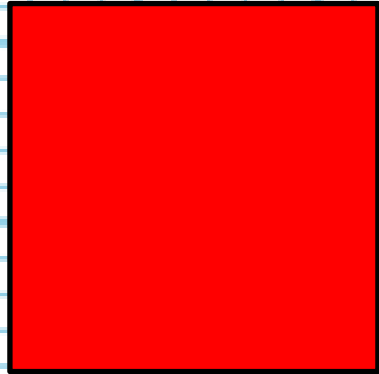
$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$

You should do some combining of polynomials with students before beginning multiplication with algebra tiles as they will need to understand what the tiles mean and how to combine like terms, terms of the same shape or power.



Algebra Tiles: Addition

$(1x^2 + 2x + 3) + (2x^2 + 3x + 4)$



$$\begin{array}{r} 1(x^2) + 2(x) + 3(1) \\ + 2(x^2) + 3(x) + 4(1) \\ \hline \end{array}$$

$$3(x^2) + 5(x) + 7(1)$$

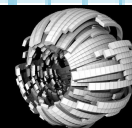
$$3x^2 + 5x + 7$$

Have them put the same shapes together, aka combine like terms.

Can you put the terms together?

No, they are not the same shape.

Squares,
Lines & Dots
or Quadratic
Terms, Linear
Terms &
Constants



Cognitive Demand & Access

“My students aren’t ready for algebra.”

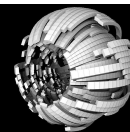
“They have to master basic skills first before learning algebra.”

While it is preferred to see algebra as a generalization of any base, could we use algebra to remediate number? Instead of going specific to general could we go general to specific?

Keep students on grade level and remediate number with Tier Two interventions using more time to relate number to algebra. In class, use polynomial addition & subtraction to remediate integers or even whole numbers.

Remediate in context of grade level material.

There is no regrouping with algebra tiles as you do not know what the base is, or how many units it takes to make the units of the next power. The lack of regrouping actually makes the cognitive demand for learning algebra less than the cognitive demand for number when using algebra tiles and base ten blocks.



Next Level of Abstraction: Drawing the Tiles Under the Expressions

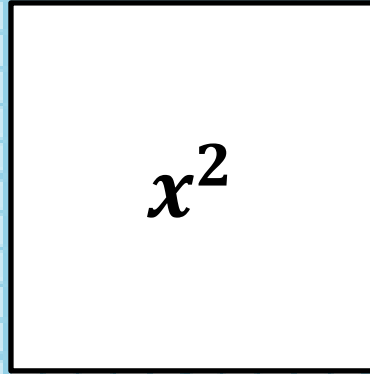
1



x



x^2



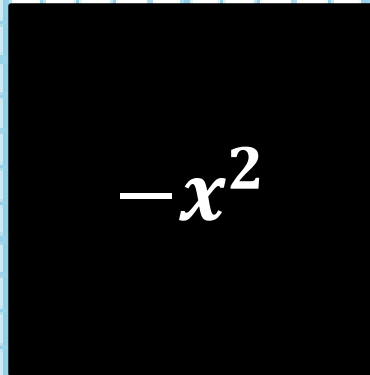
-1



$-x$



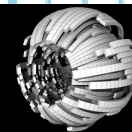
$-x^2$



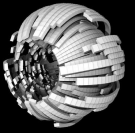
“Open” is positive.
“Closed” is negative.







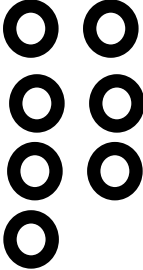
Referring to them as white being positive and black being negative is not a good idea.

For a full discussion of polynomial concreteness fading including subtraction. See the slides from the morning session posted at [meaning for memory.com](http://meaningformemory.com)



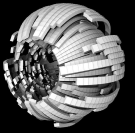
Combining Expressions: Three Column Notes

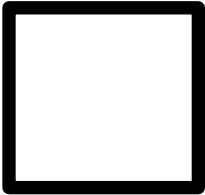





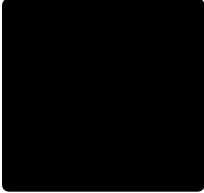


	+		=	
$1x^2 - 2x + 3$		$-2x^2 - 3x + 4$		$-1x^2 - 5x + 7$
				
				

Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

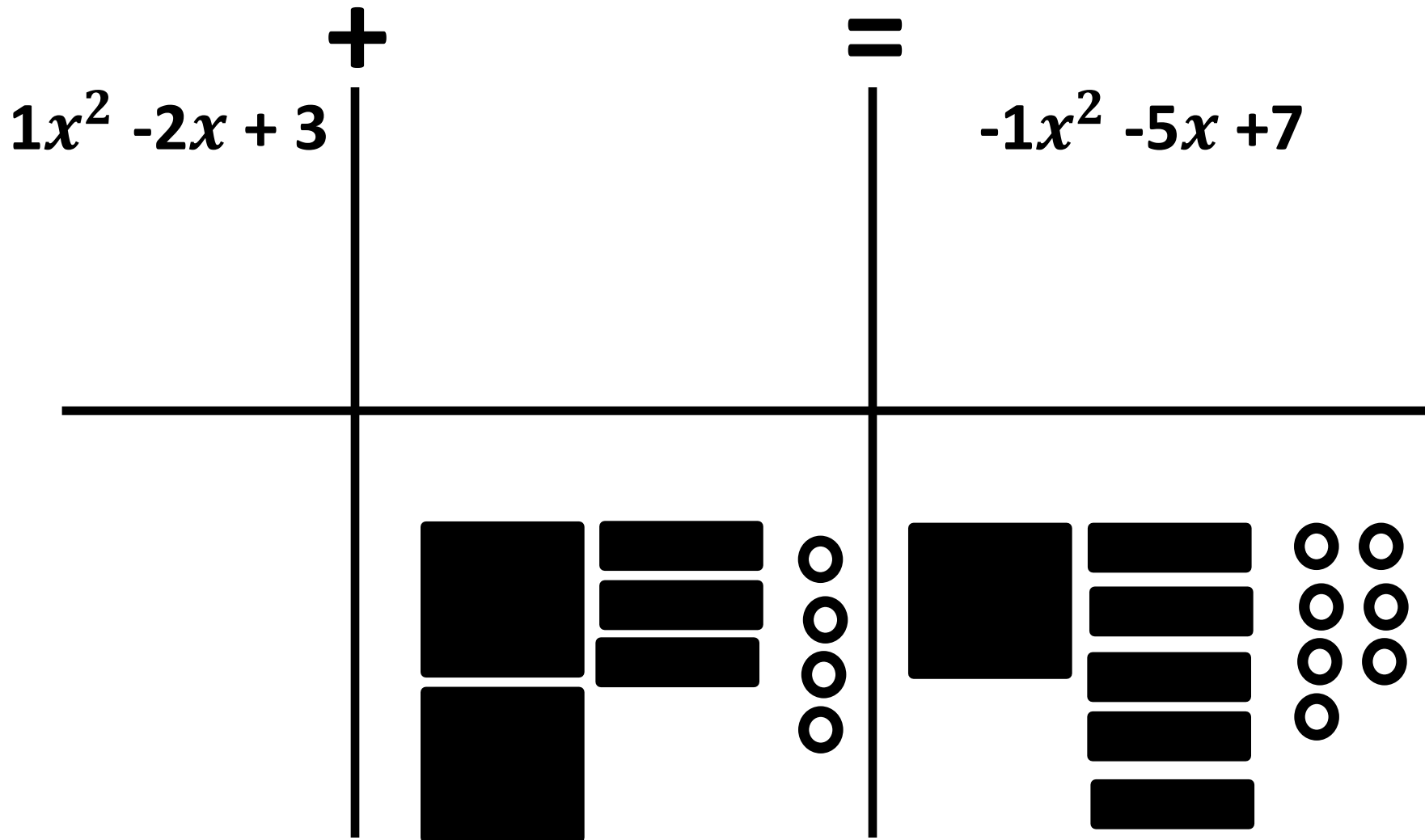
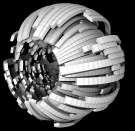


	+		=
$1x^2 - 2x + 3$		$-2x^2 - 3x + 4$	
<hr/>			
			
			
			

Eliminate just enough information to make it doable.

Require students to draw the shapes AND write the symbols.

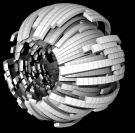
Combining Expressions: Three Column Notes



Eliminate just enough information to make it doable.

Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes

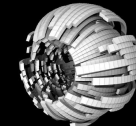


$+$	$=$	$-1x^2 - 5x + 7$
		<div><div></div><div></div><div></div><div></div><div></div></div> <div><div></div><div></div><div></div><div></div></div>

Students make their own problem.

Require students to draw the shapes AND write the symbols.

Combining Expressions: Three Column Notes



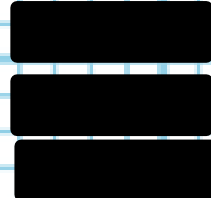
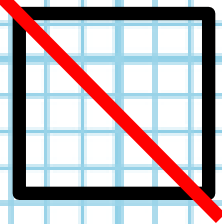
1x² - 2x + 3 **-** **-2x² - 3x + 4** **=** **3x² + 1x - 1**

If you do subtraction, then make them change the shapes to the opposite before combining.

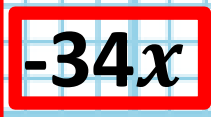
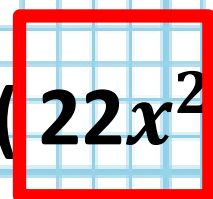
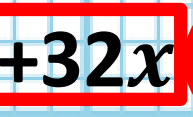
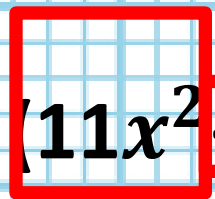
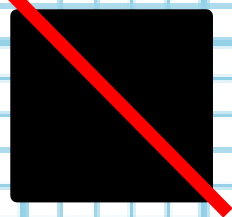
The diagram shows the subtraction of the second polynomial from the first. The first polynomial is represented by one large square (x²), two horizontal rectangles (x), and three small circles (1). The second polynomial is represented by two large squares (x²), three horizontal rectangles (x), and four small circles (1). A blue arrow points from the second polynomial's tiles to the first polynomial's tiles, indicating the subtraction process. The result is represented by three large squares (x²), one horizontal rectangle (x), and one small circle (1).

Next Level of Abstraction: Drawing the Tiles Around the Symbols

$$(1x^2 - 2x + 3) + (-2x^2 - 3x + 4)$$



$$-1x^2 - 5x + 7$$



$$(11x^2 + 32x + 13) + (22x^2 - 34x + 14)$$

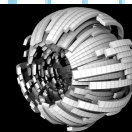
$$33x^2 - 2x + 27$$

This is a good time to introduce larger coefficients.

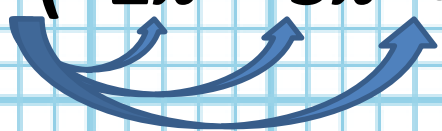
THE HORIZONTAL METHOD IS GOOD FOR DRAWING SHAPES AROUND OR BELOW.

Make your squares, quadratic, large and your rectangles, linear, smaller, but with the same side length.

Constants get circles instead of the little squares that are more difficult to draw and easily confused with quadratics.



Final Level of Abstraction: Line Up Terms Vertically

$$\begin{array}{r} (1x^2 - 2x + 3) - (-2x^2 - 3x + 4) \\ +2x^2 + 3x - 4 \\ \hline 3x^2 + 1x - 1 \end{array}$$


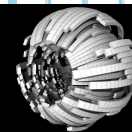
Keep asking, "Can I simplify?"

THE VERTICAL METHOD IS GOOD FOR KEEPING LIKE TERMS

TOGETHER AND REWRITING SUBTRACTION.

You can reference the shapes as you do it . You want students to follow the procedure of combining like terms based on what they learned about the areas they represent.

You can redraw them at the end to reinforce it.



Final Level of Abstraction: Line Up Terms Vertically

$$\begin{array}{r} (1x^2 - 2x + 3) - (-2x^2 - 3x + 4) \\ +2x^2 + 3x - 4 \\ \hline 3x^2 + 1x - 1 \end{array}$$

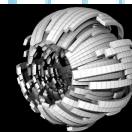
No, quadratic, linear & constant

THE VERTICAL METHOD IS GOOD FOR KEEPING LIKE TERMS

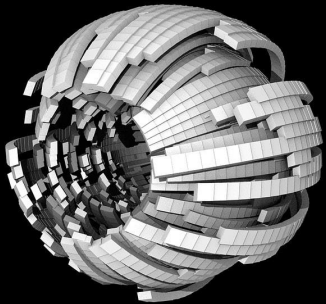
TOGETHER AND REWRITING SUBTRACTION.

You can reference the shapes as you do it . You want students to follow the procedure of combining like terms based on what they learned about the areas they represent.

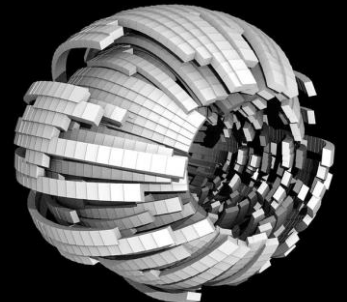
You can redraw them at the end to reinforce it.



Polynomial Multiplication: Area & Partial Products



David Mattoon
Meaning for Memory



Lesson Introduction: Race the Calculator



$$\begin{array}{r} 22 \\ \times 43 \\ \hline 66 \\ 880 \\ \hline 946 \end{array}$$

$$\begin{array}{r} 22 \\ \times 43 \\ \hline 800 \end{array} \quad \begin{array}{r} 20 + 2 \\ \times 40 + 3 \\ \hline 800 \end{array}$$

80

80

60

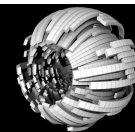
60

6

6

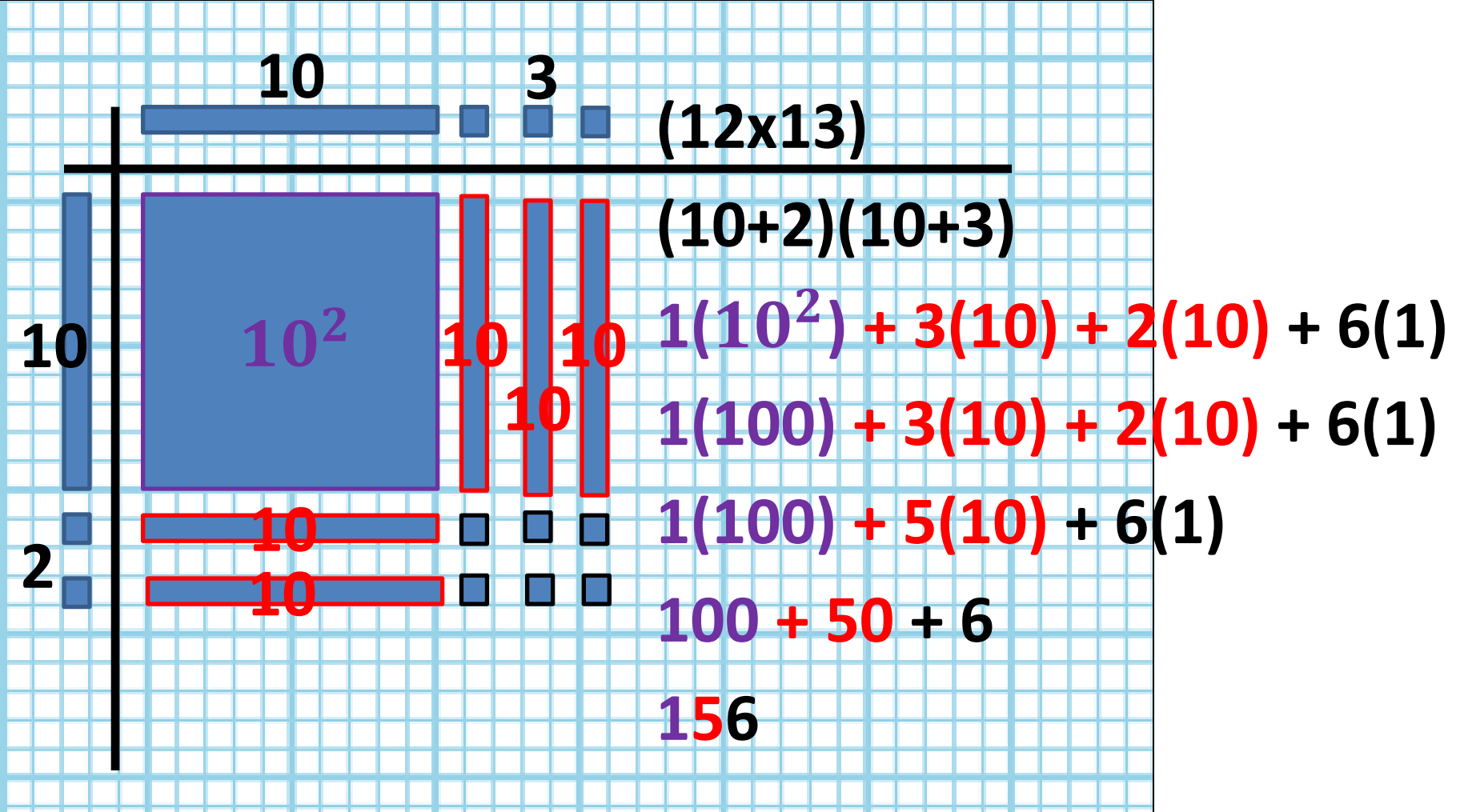
946

946



Find the Product of 12×13 Using Base Ten Blocks

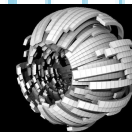
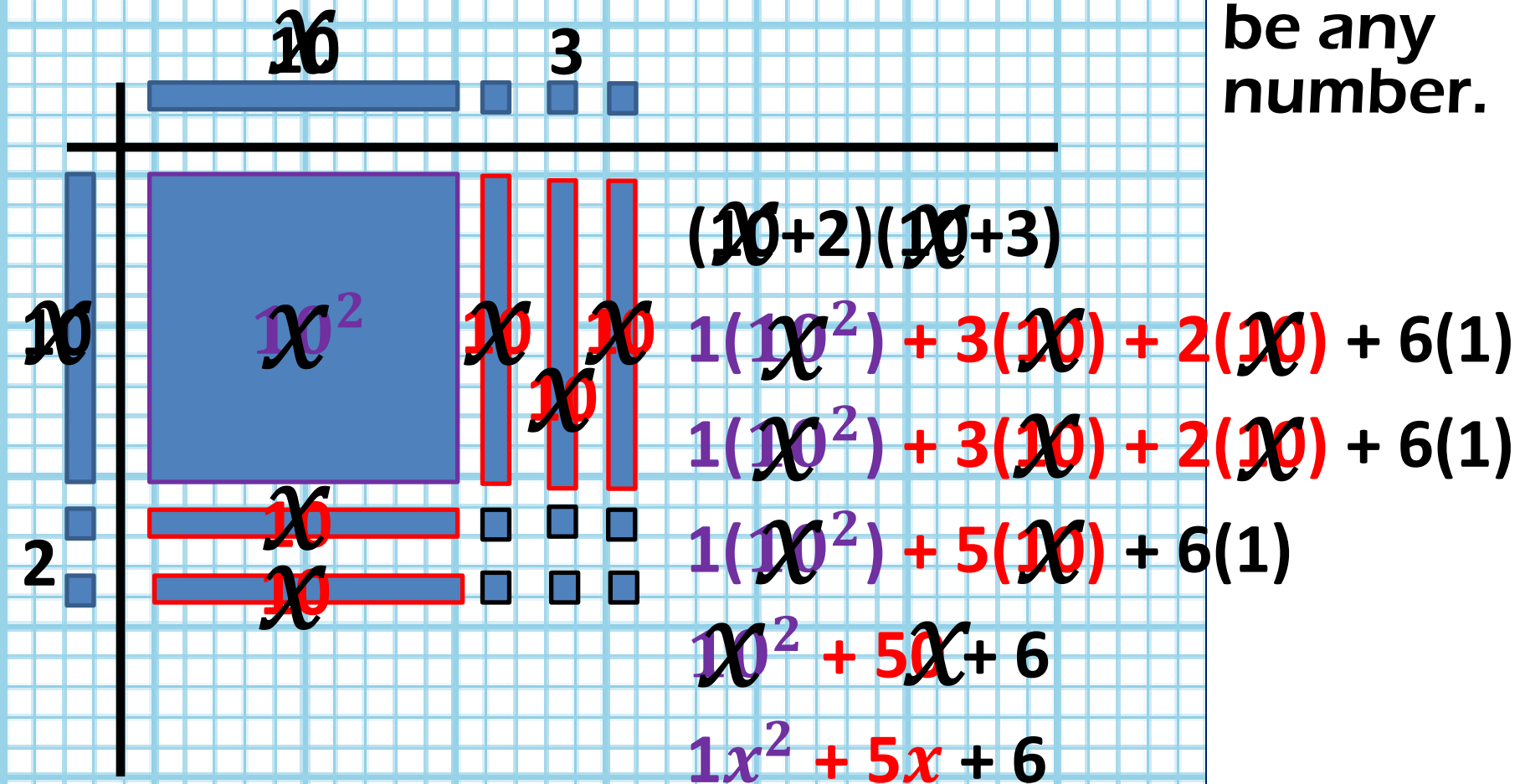
Start with
base ten.



Transition to Algebra Tiles

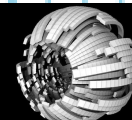
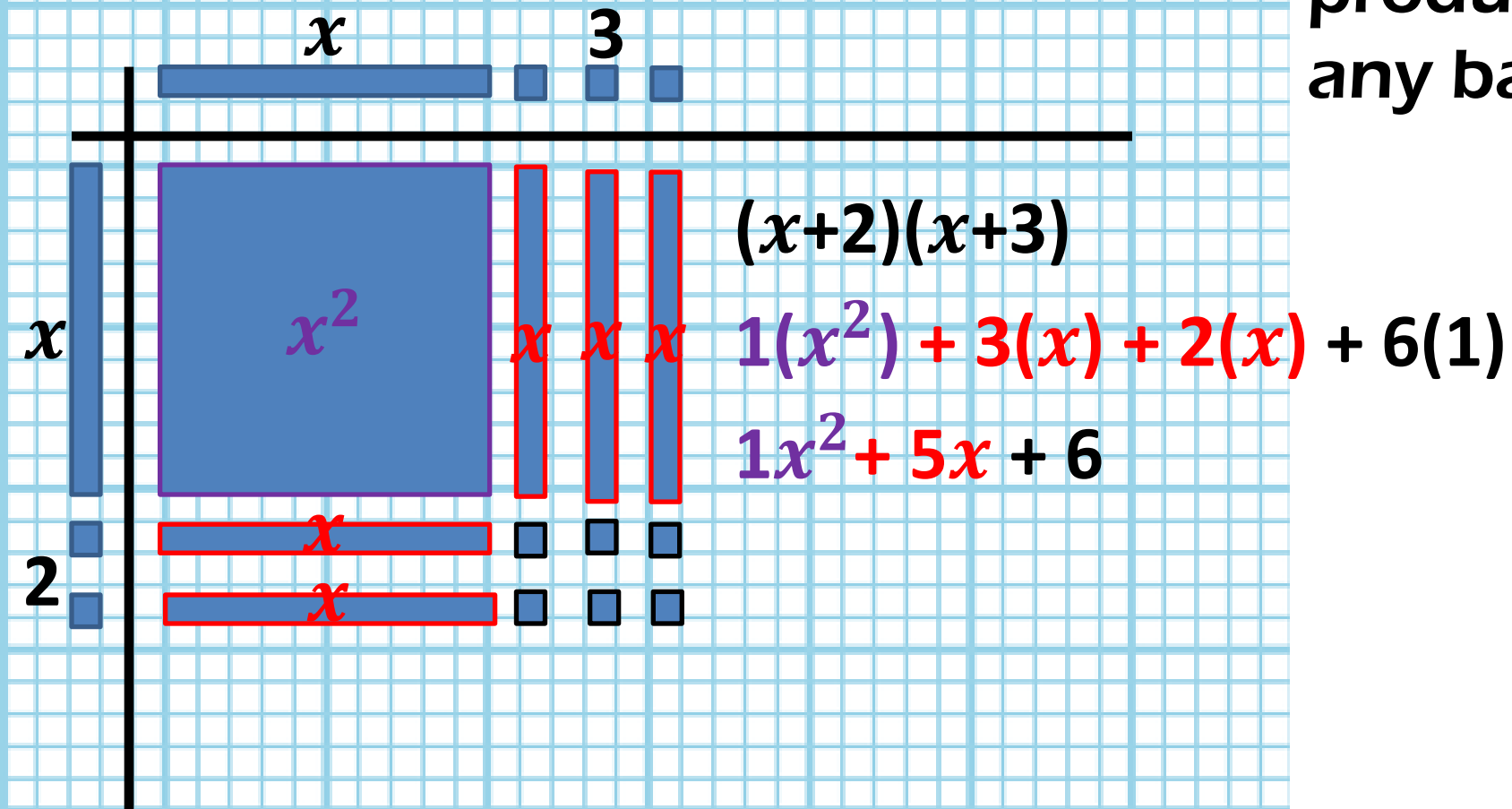
I hate the number ten!

**I want it to
be any
number.**



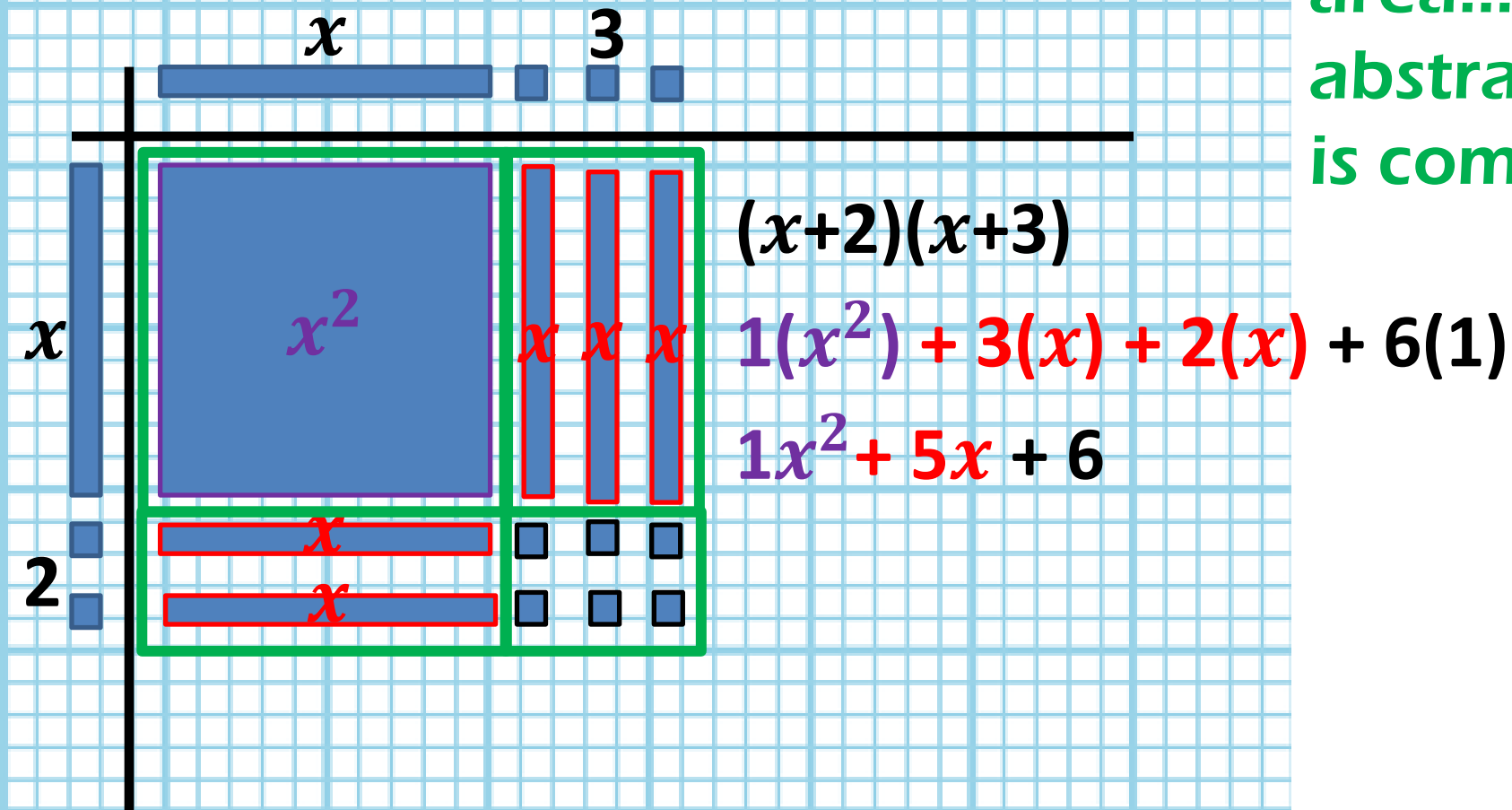
Find the Product of $(x + 2)(x + 3)$ Using Algebra Tiles

**You are
now
finding the
product of
any base.**

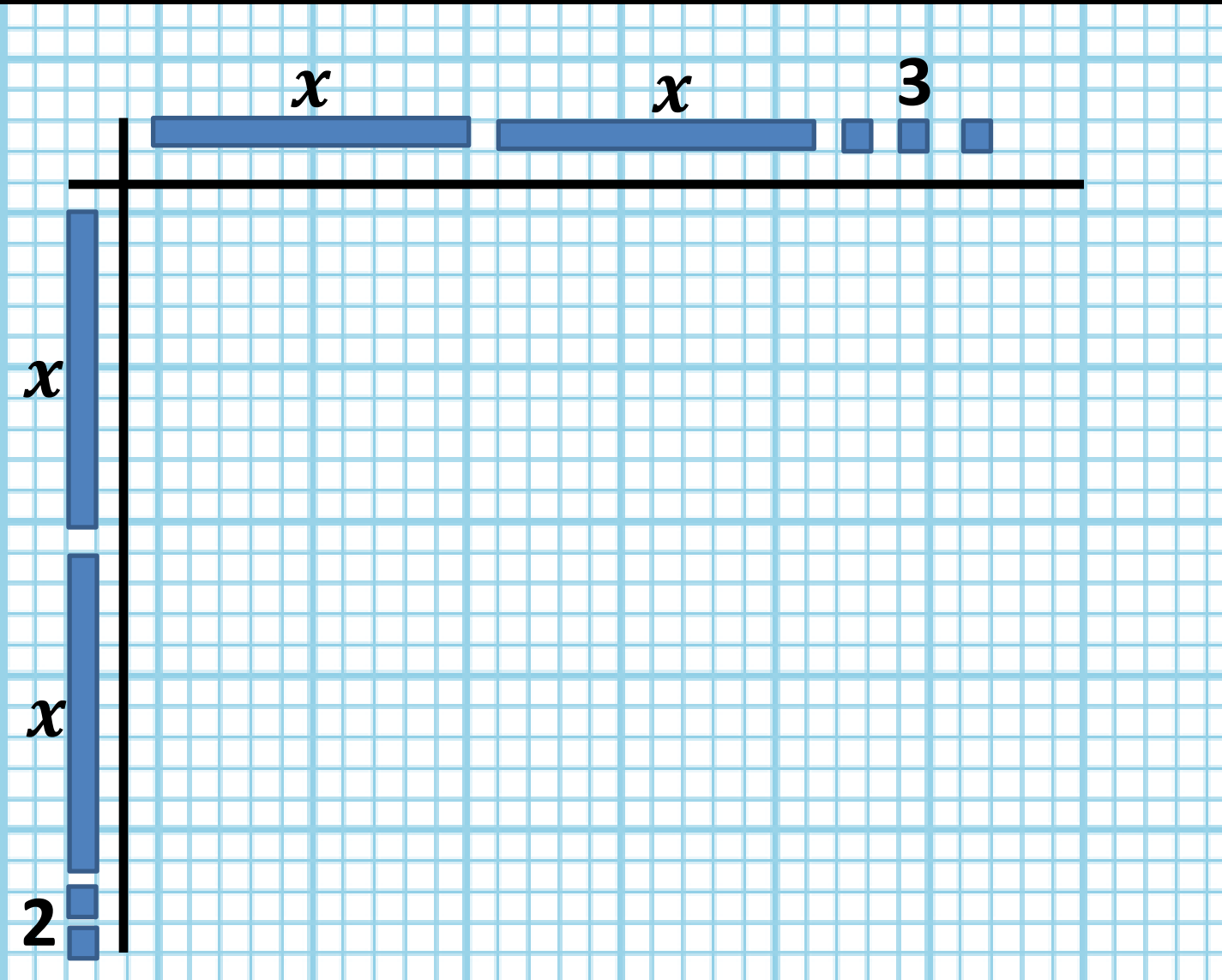
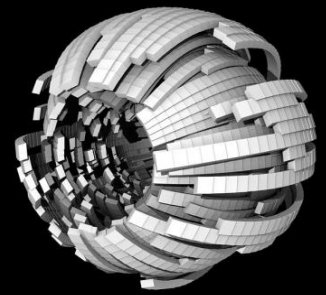


Algebra Tiles: Working Toward the Next Level of Abstraction

Notice the
four
regions of
area...
abstraction
is coming...



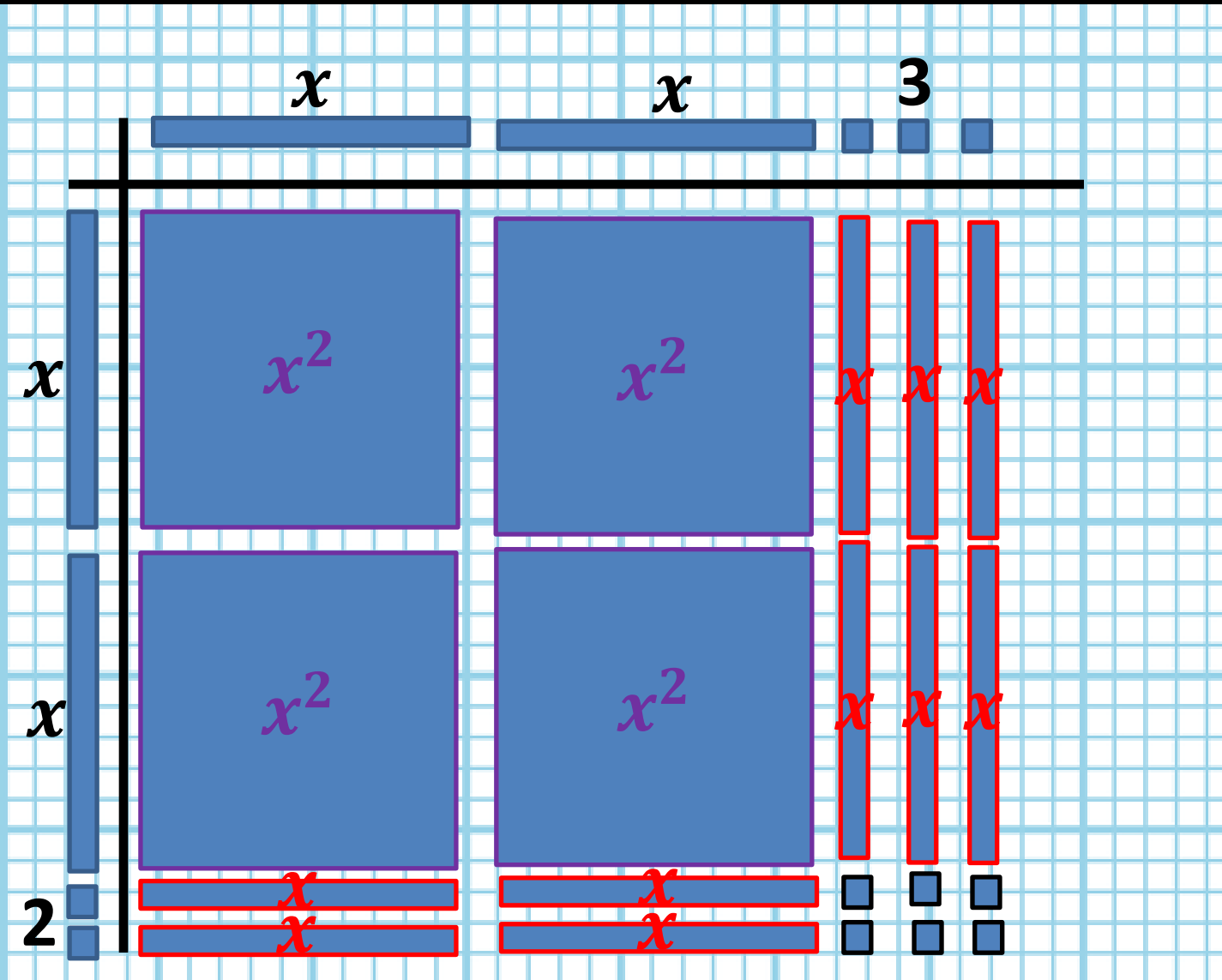
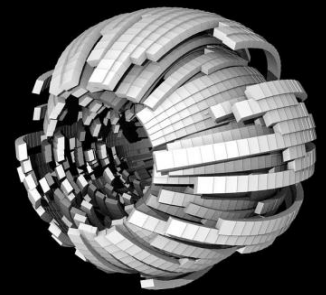
Try One: Find the Product of $(2x + 2)(2x + 3)$ Using Algebra Tiles



Use your whiteboard as a frame. Put the side lengths on the outside edge off the board and the product on the board.

Write your answer as a trinomial on your whiteboard, then check your answer with a partner.

Try One: Find the Product of $(2x + 2)(2x + 3)$ Using Algebra Tiles

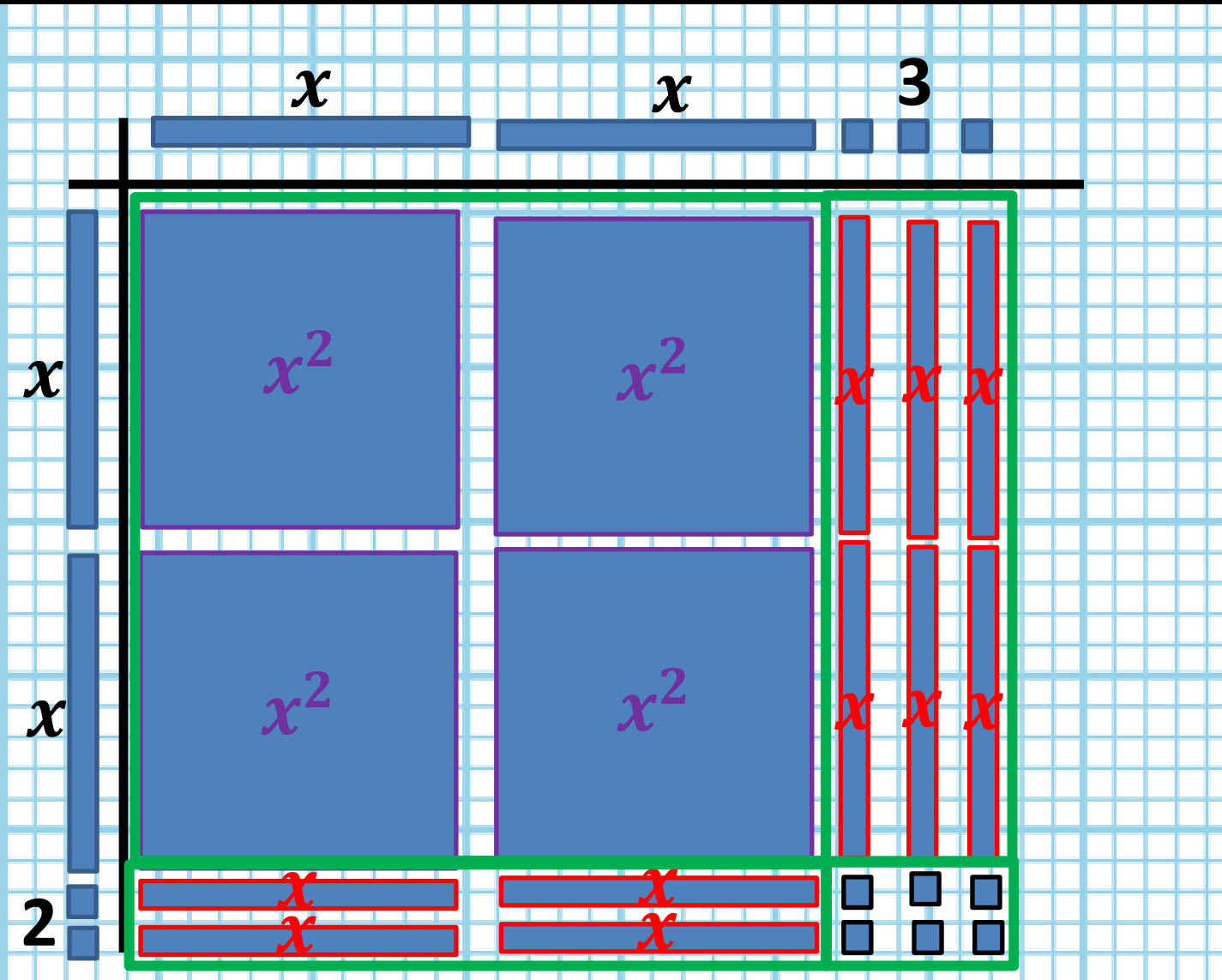
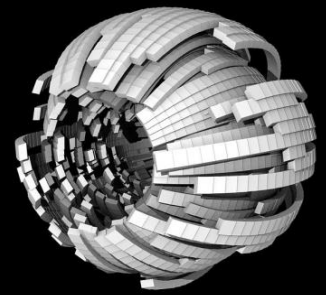


$$(2x+2)(2x+3)$$

$$4(x^2) + 6(x) + 4(x) + 6(1)$$

$$4x^2 + 10x + 6$$

Try One: Find the Product of $(2x + 2)(2x + 3)$ Using Algebra Tiles



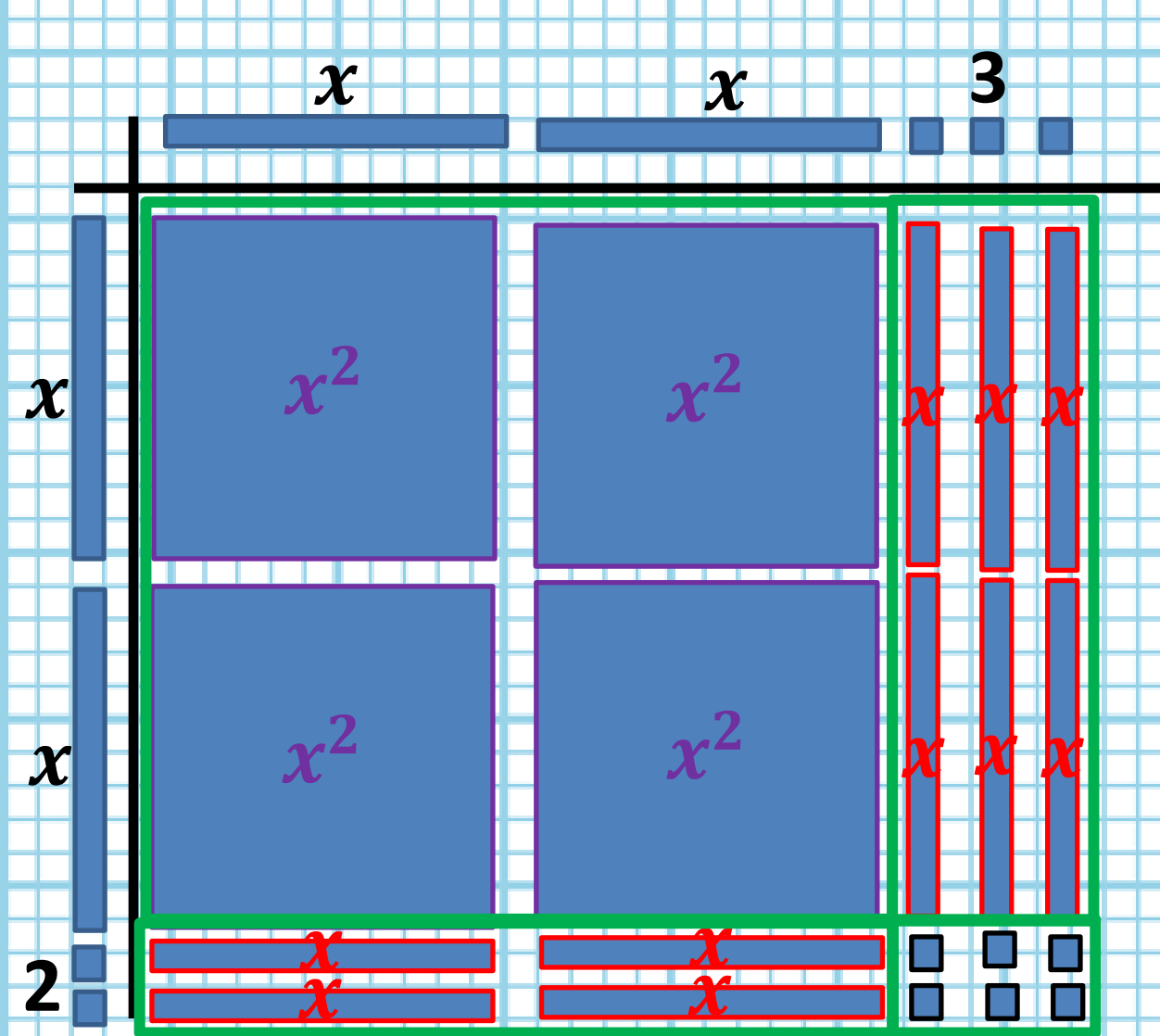
Notice the
 four
 regions of
 area...
 abstraction
 is coming...

$$(2x+2)(2x+3)$$

$$4(x^2) + 6(x) + 4(x) + 6(1)$$

$$4x^2 + 10x + 6$$

The Effect of Standard Form



From the two examples we have done:

What do you notice about the quadratic (square) term in the picture?

It is always on the _____.

What do you notice about the linear terms in the picture? They are always on the _____.

What do you notice about the constant term in the picture?

It is always on the _____.

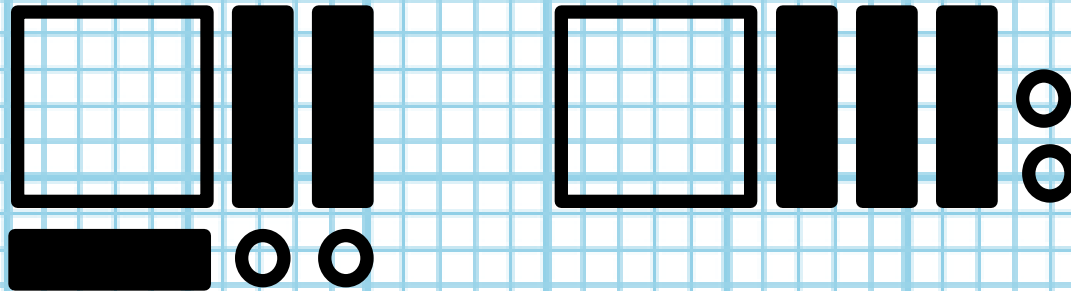
Word Bank:

Top Bottom

Left Right

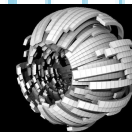
Diagonal

Next Level of Abstraction: Draw the Shapes



Open terms
are positive
and closed
terms are
negative.

Referring to
them as white
being positive
and black
being
negative is
not a good
idea.



Multiplying Polynomials: Four Column Notes

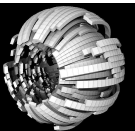
Fold Your Paper into Four Columns

Factored
Expression

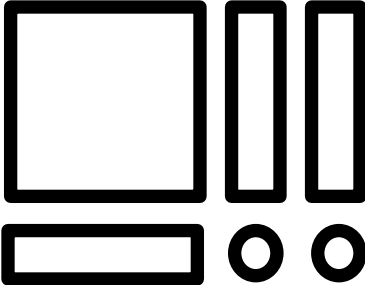
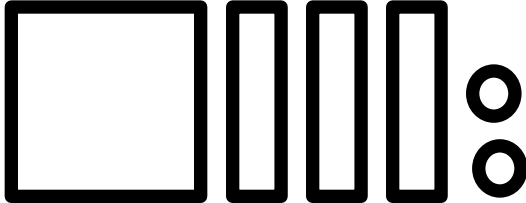
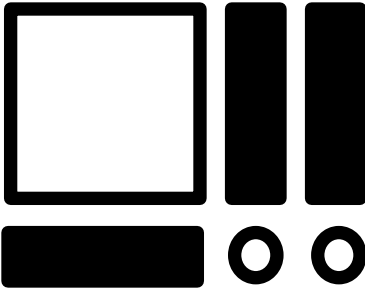

Factored
Pictorial

Unfactored
Pictorial

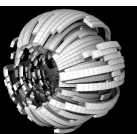
Unfactored
Expression



All Four Representations to Build Understanding

<u>Factored Expression</u>	<u>Factored Pictorial</u>	<u>Unfactored Pictorial</u>	<u>Unfactored Expression</u>
$(x + 2)(x + 1)$			$x^2 + 3x + 2$
$(x - 2)(x - 1)$			$x^2 - 3x + 2$

I included a negative term here as an example if you choose to go there before higher levels of abstraction.



Move Fluidly between Representations

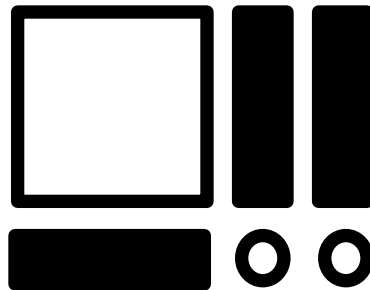
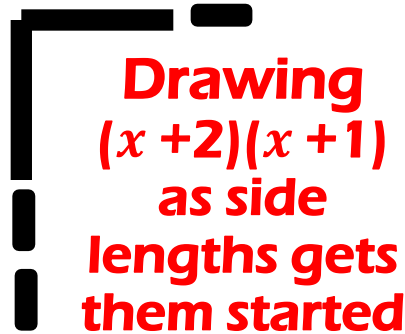
Factored
Expression

Factored
Pictorial

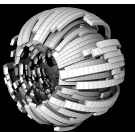
Unfactored
Pictorial

Unfactored
Expression

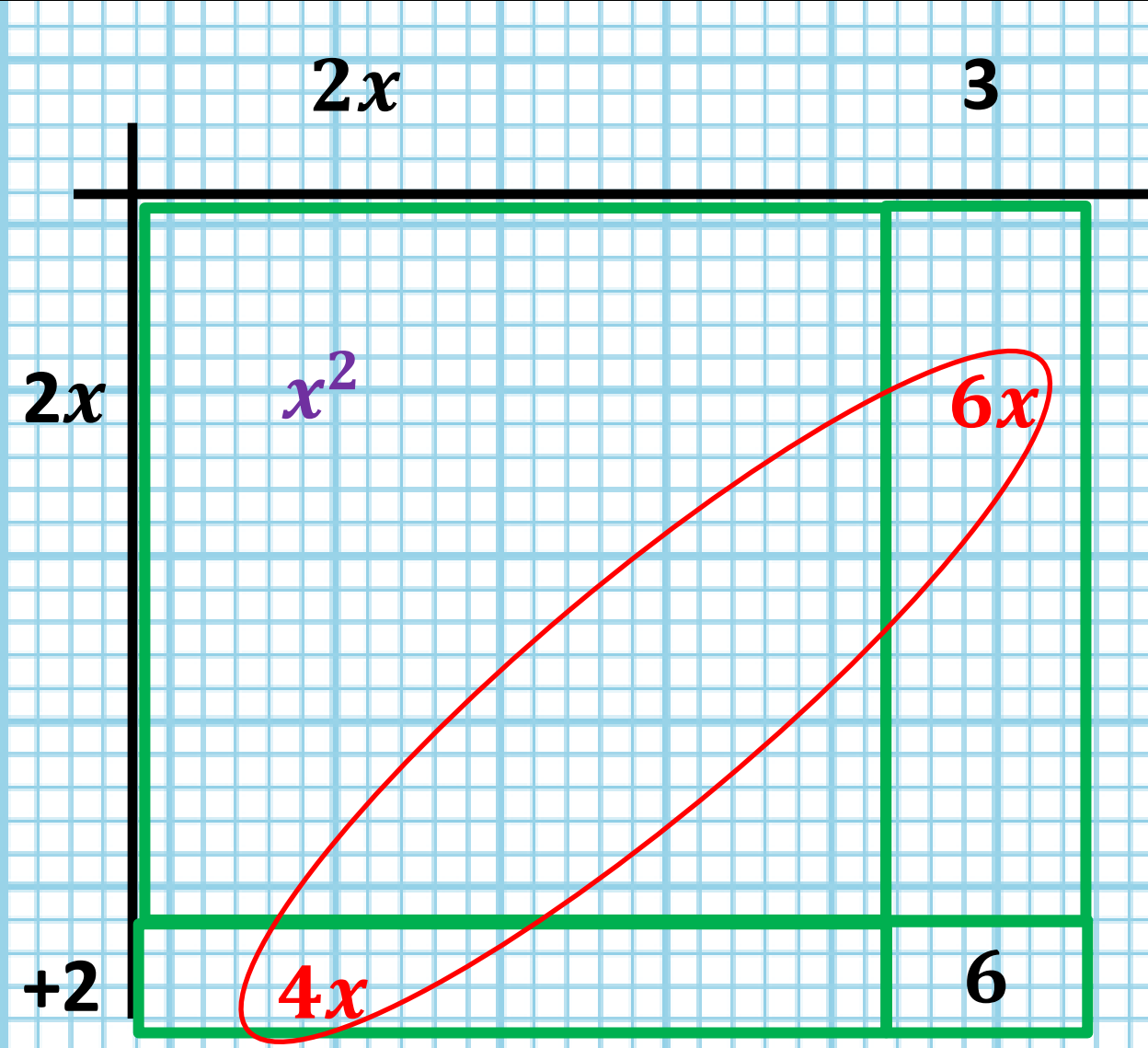
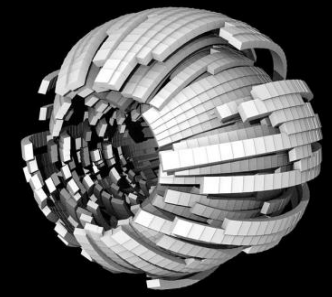
$$(x + 2)(x + 1)$$



Once they have practiced, you can give them a single expression and ask for the others.



Next Level of Abstraction Area Representation without the Tiles (to scale)



I rarely use this abstraction step; however, I might show it once to students.

Box Method?

The terms represent the tiles or the area of the rectangle they are within, which needs to be reinforced. They are not merely entries in a table or box.

Still $(2x+2)(2x+3)$

$4(x^2) + 6(x) + 4(x) + 6(1)$

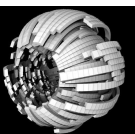
$4x^2 + 10x + 6$

Emphasize combining like terms on the diagonal

Table, Chart or Boxes?

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

**Tell someone next you
what you think.**



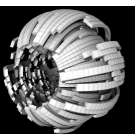
Table, Chart or Boxes?

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

Trick Question:

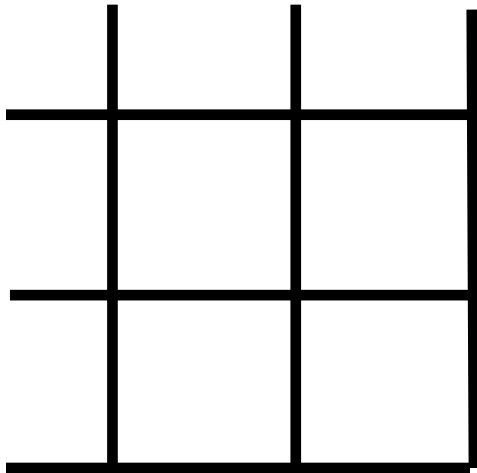
**None of the above,
AREA!**

**Check Out (11)(12) or
(10+1)(10+2)
100+30+2
132**

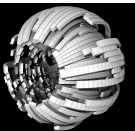


Next Level of Abstraction: Box Method

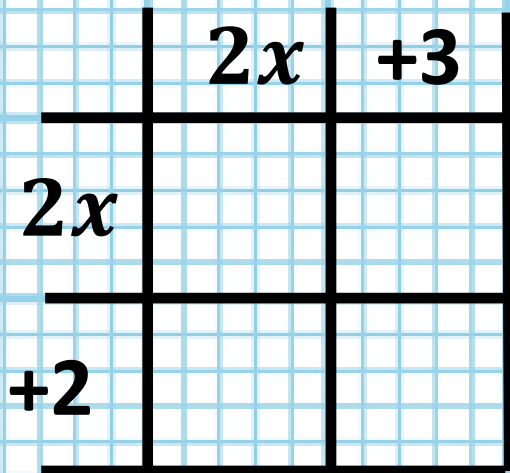
The box method is a level of abstraction of the area method. Taught without the algebra tiles it means no more than numbers in a box like a multiplication table.



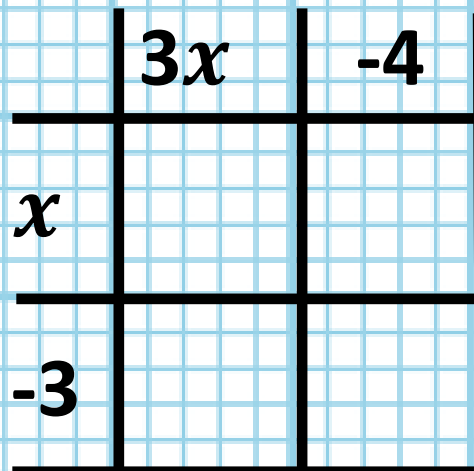
Algebra tiles should be done first so when students use the Xbox they visualize the algebra tiles. Thus, avoiding the common mistake of FL instead of FOIL, which by the way is next to useless besides connecting to parents. FOIL only works for binomial multiplication. The distributive property is what we are working toward.



Next Level of Abstraction Area Representation without the Tiles (not to scale)



$$(2x+2)(2x+3)$$

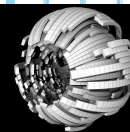


$$(3x-4)(x-3)$$

This is a good time to bring back negatives if you have skipped them for the more conceptual pieces.

By the way, negatives are effective for base 10 and mental multiplication like $(50-1)(50-1)$ instead of $(49)(49)$.

I like to extend the lines outside of the large, exterior rectangle to help students see negative signs, and it makes them resemble the tick marks indicating length.



Next Level of Abstraction Area Representation without the Tiles (not to scale)

	$2x$	$+3$
$2x$	$4x^2$	$6x$
$+2$	$4x$	6

$$(2x+2)(2x+3)$$

$$4x^2 + 6x + 4x + 6$$

$$4x^2 + 10x + 6$$

	$3x$	-4
x	$3x^2$	$-4x$
-3	$-9x$	12

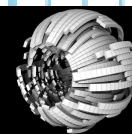
$$(3x-4)(x-3)$$

$$3x^2 - 4x - 9x + 12$$

$$3x^2 - 13x + 12$$

You are using the side lengths to find the area of each rectangle (partial products). Emphasize this with students.

Continue to emphasize combining like terms on the diagonal.



Try One:

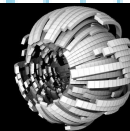
Multiplying Two Trinomials

	$6x^2$	$+5x$	$+4$
$3x^2$			
$+2x$			
$+1$			

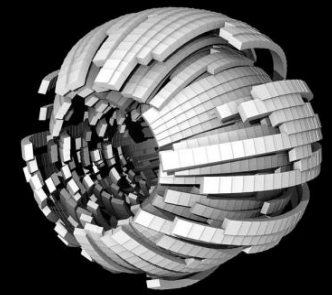
$$(6x^2 + 5x + 4)(3x^2 + 2x + 1)$$

On your whiteboard or notes, find the product.

What do you notice about the like terms?



Order In... Order Out



	$6x^2$	$5x$	4	
$3x^2$	$18x^4$	$15x^3$	$12x^2$	
$2x$	$12x^3$	$10x^2$	$8x$	
$+1$	$6x^2$	$5x$	4	

$$27x^3$$

$$28x^2$$

$$13x$$

$$(6x^2 + 5x + 4)(3x^2 + 2x + 1)$$

$$18x^4 + 12x^3 + 15x^3 + 6x^2 + 10x^2 + 12x^2 + 5x + 8x + 4$$

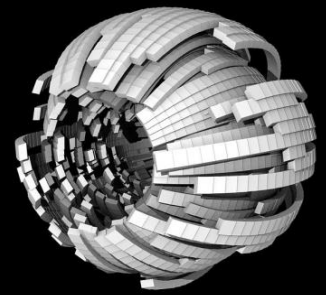
$$18x^4 + 27x^3 + 28x^2 + 13x + 4$$

When you fill out the lengths using standard form, then the like terms are on the diagonals.

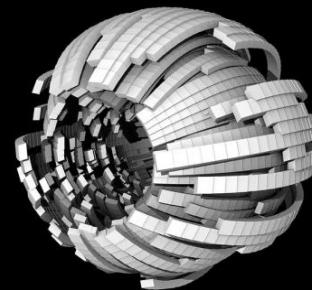
Depending on the polynomial this is not always true, but when it is it really helps avoid mistakes with mislabeling, especially for more complex problems.

Do a problem like $(6x^2 + 5x + 4)(3x^2 + 1)$ to show students that it is not always the case.

Time: I don't have it.



Time Gained



	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$

$$(3x^2)(6x^2 + 5x + 4)$$

$$18x^4 + 15x^3 + 12x^2$$

Ask, "how many terms in the first factor?" and "How many terms in the second factor?" in order to draw the correct "box."

	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$
$2x$	$12x^3$	$10x^2$	$8x$

$$27x^3 \quad 22x^2$$

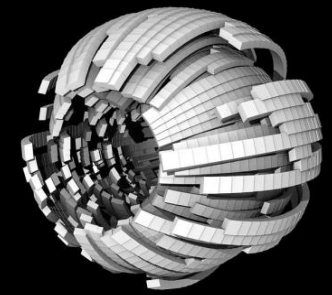
$$(3x^2 + 2x)(6x^2 + 5x + 4)$$

$$18x^4 + 27x^3 + 22x^2 + 8x$$

Time spent on algebra tiles is recouped by a single method used for multiple problems.

More time is recouped by less time reviewing as there is only one method to remember in the first place or one method to review with students.

Remediation & SpEd



	$6x^2$	$5x$	4	
$3x^2$	$18x^4$	$15x^3$	$12x^2$	
$2x$	$12x^3$	$10x^2$	$8x$	
$+1$	$6x^2$	$5x$	4	

$$27x^3$$

$$28x^2$$

$$13x$$

$$(6x^2 + 5x + 4)(3x^2 + 2x + 1)$$

$$18x^4 + 12x^3 + 15x^3 + 6x^2 + 10x^2 + 12x^2 + 5x + 8x + 4$$

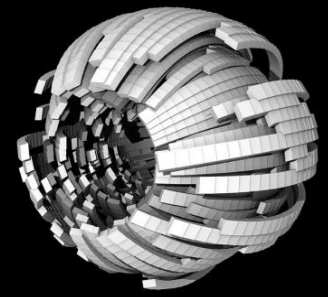
$$18x^4 + 27x^3 + 28x^2 + 13x + 4$$

Need to review math facts? Why not teach polynomials? Multiplication, addition, and subtraction through negatives.

Division? Give them the product and a factor and ask for the other factor, length of the unknown side.

Final Level of Abstraction

The Distributive Property



	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$

$$(3x^2)(6x^2 + 5x + 4)$$

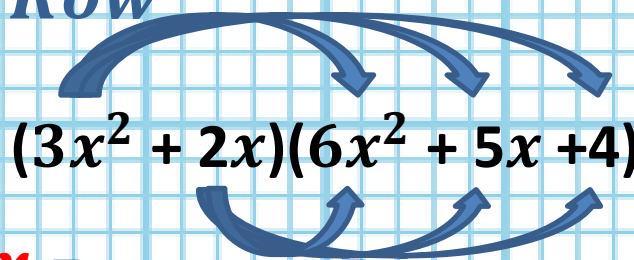


$$18x^4 + 15x^3 + 12x^2 \text{ Top Row}$$

$$18x^4 + 15x^3 + 12x^2$$

$$12x^3 + 10x^2 + 8x \text{ Bottom Row}$$

	$6x^2$	$5x$	4
$3x^2$	$18x^4$	$15x^3$	$12x^2$
$2x$	$12x^3$	$10x^2$	$8x$



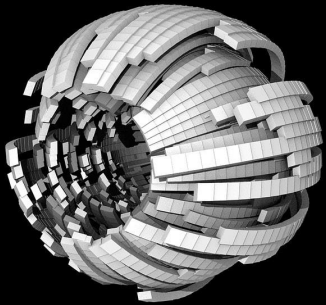
$$(3x^2 + 2x)(6x^2 + 5x + 4)$$

$$18x^4 + 27x^3 + 22x^2 + 8x$$

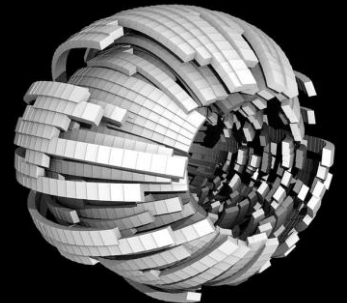
Students should be able visualize or use their experience with the area representation and/or the box to distribute correctly.

Line up like terms vertically like before to get the result in the second example.

Polynomial Factoring: Making Rectangles & Finding Side Lengths



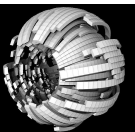
David Mattoon
Meaning for Memory



Warm-up

Using only a
guess and check method
solve for x:

$$2x^2 + 7x + 6 = 0$$



Why are we even factoring?



**KEEP
CALM**

AND

**AND FEEL
THE PAIN!!**

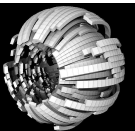
KeepCalmAndPosters.com

Did you feel the pain?

**You want them to feel the
pain so they appreciate
the Zero Product Property**

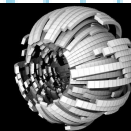
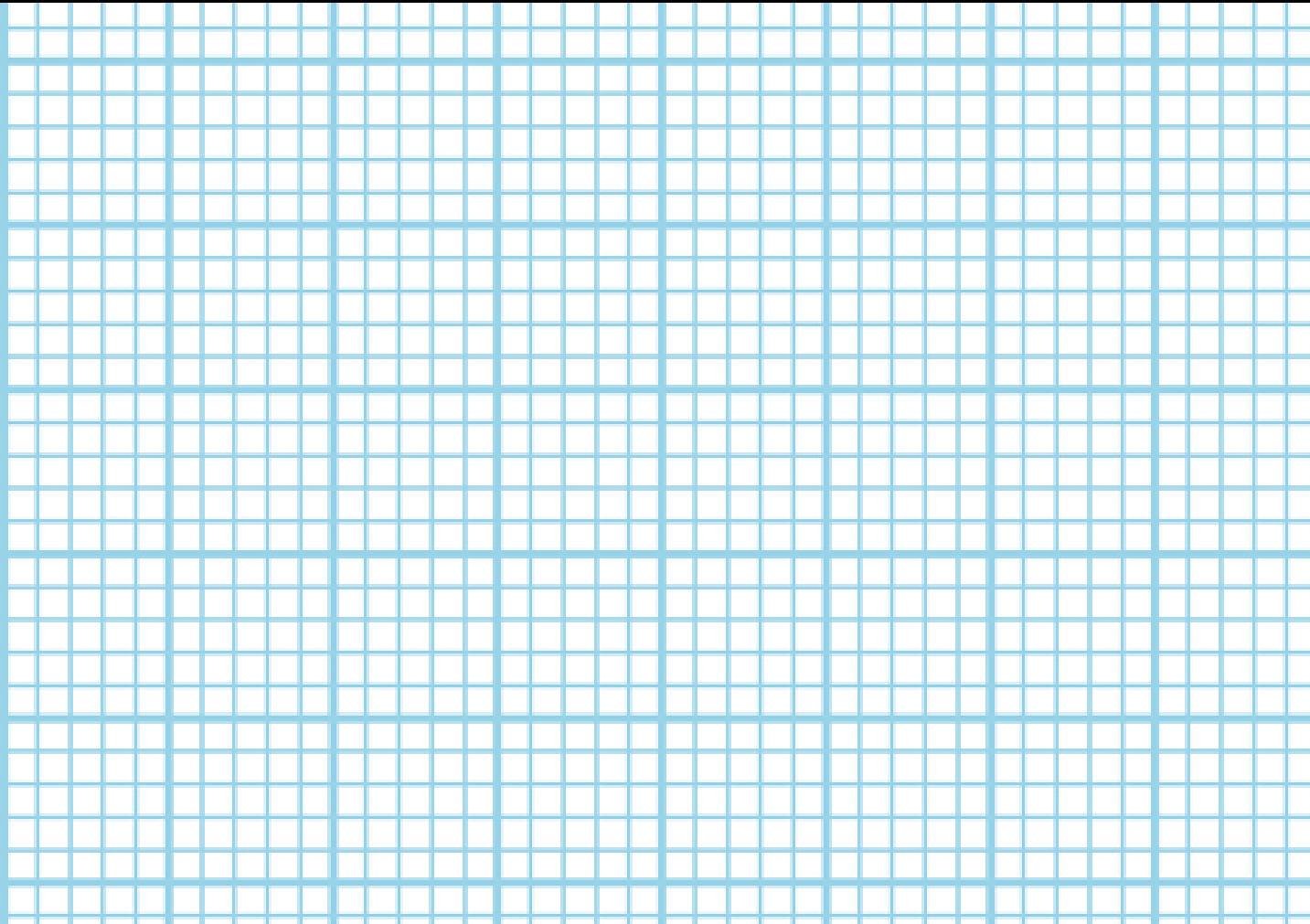
$$(2x+3)(x+2) = 0$$

But how do you get there?

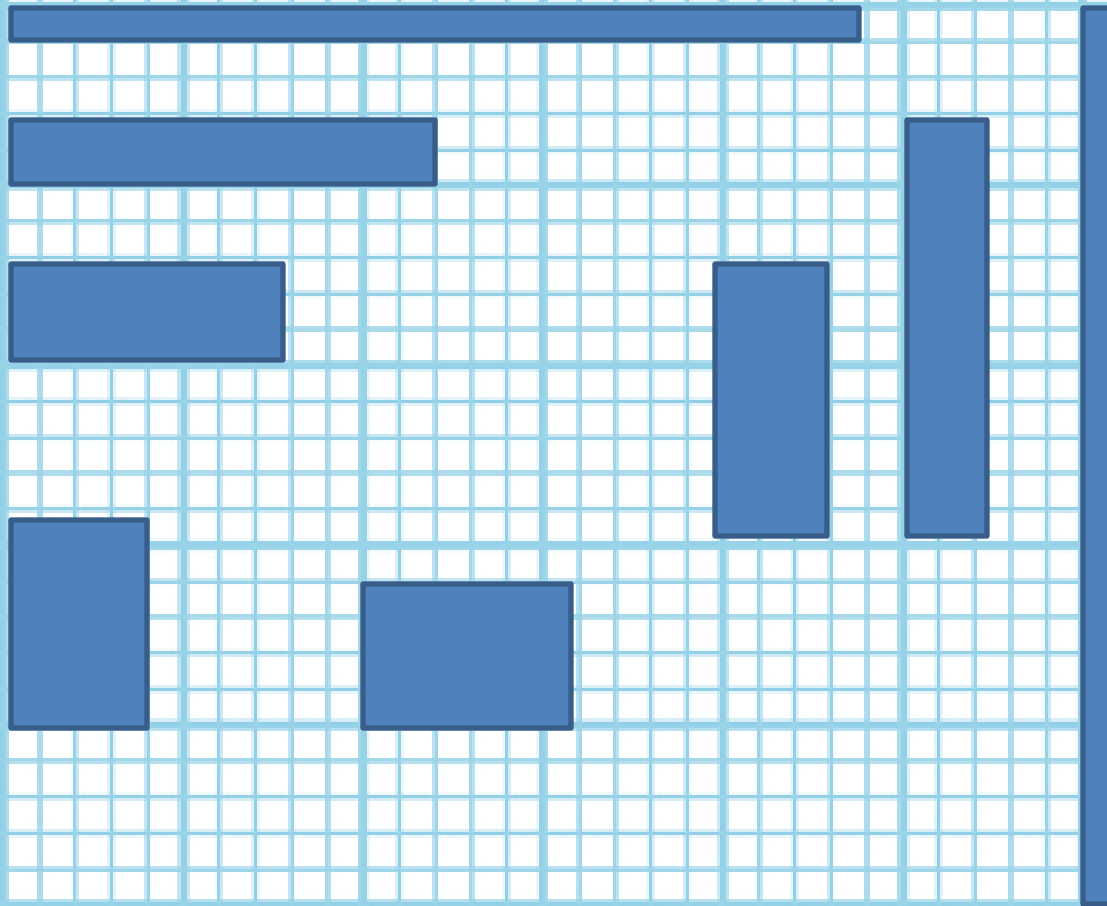


Lesson Introduction: Make Rectangles

On the gridded side of your whiteboard, make as many rectangles as you can with an area of 24 square units.



Lesson Introduction: Make Rectangles



$$1 \times 24$$

$$24 \times 1$$

$$2 \times 12$$

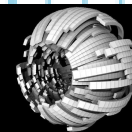
$$12 \times 2$$

$$3 \times 8$$

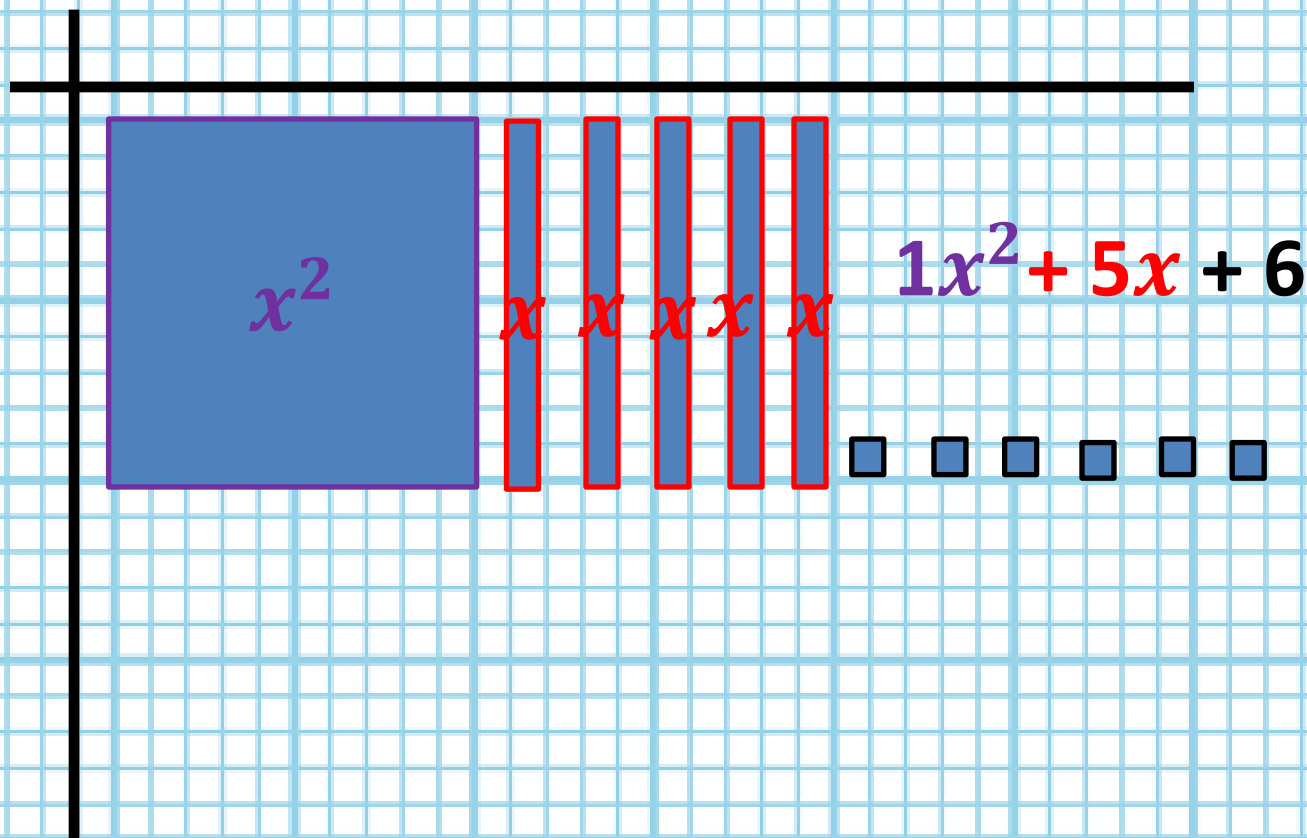
$$8 \times 3$$

$$4 \times 6$$

$$6 \times 4$$



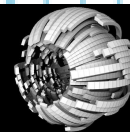
Try One: Factor $1x^2 + 5x + 6$ using Algebra Tiles



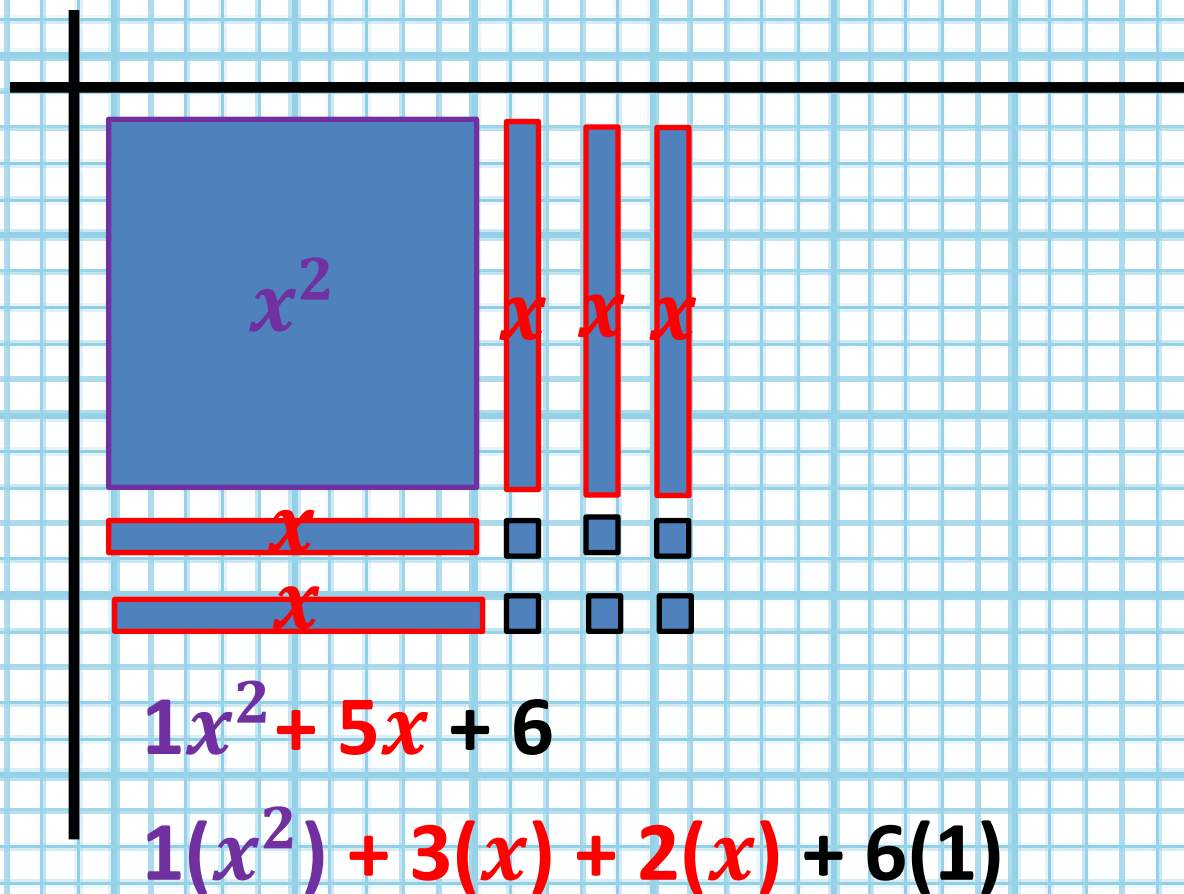
Draw the frame on your whiteboard leaving room to write in side lengths later and then factor

$$1x^2 + 5x + 6.$$

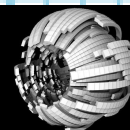
Remember when you multiplied polynomials and make the area taken up by the shapes into a rectangle.



Try One: Factor $1x^2 + 5x + 6$ Making the Rectangle

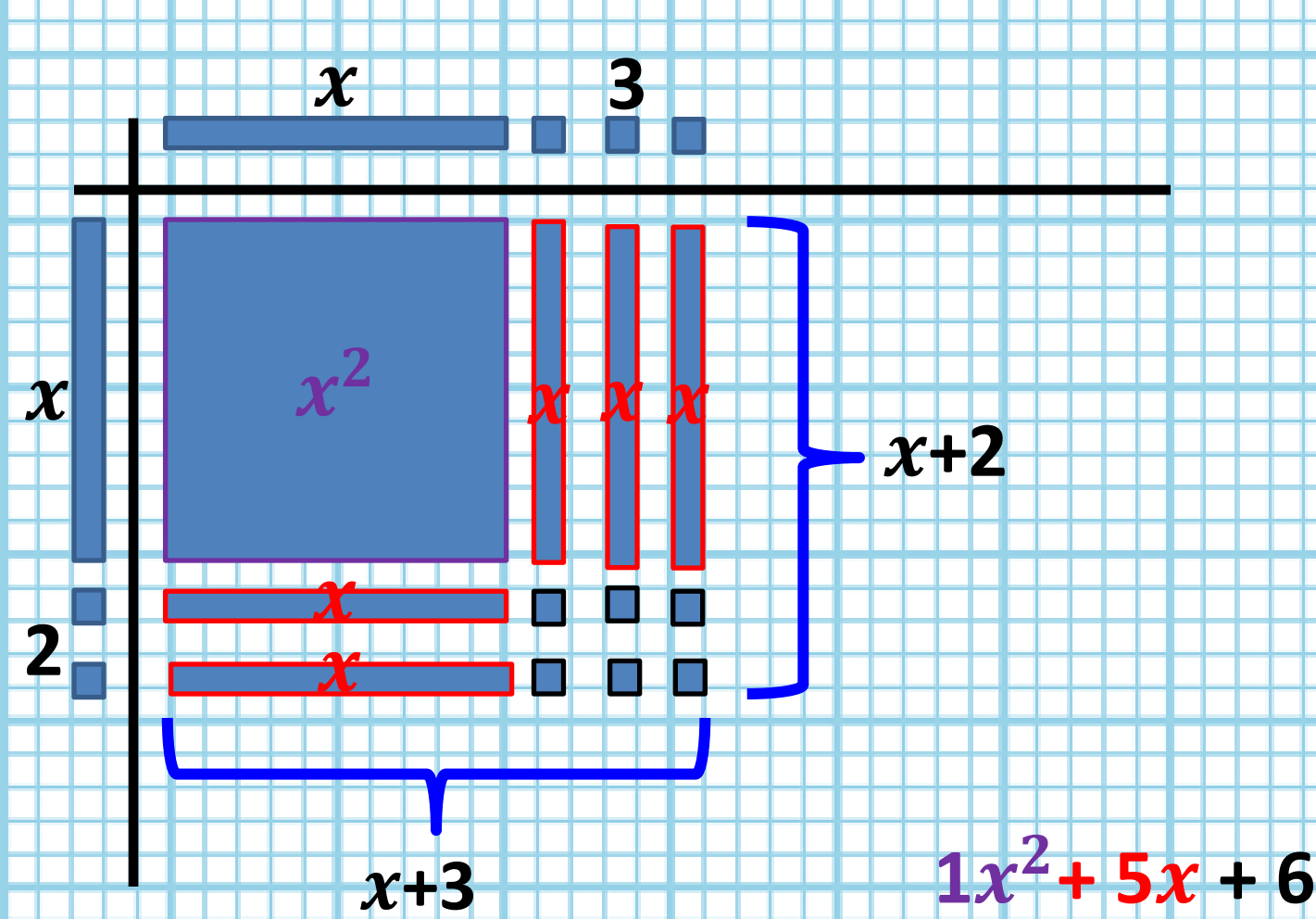


Remember the quadratics were on the top left and the constants are on the bottom right. Soon students realize the only possibilities for the sum of the linear terms are the factors of the constant, but which factors of the constant work? **Read the lengths of each side of your rectangle and write in the factors on the frame on your whiteboard.**

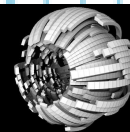


Try One: Factor $1x^2 + 5x + 6$

Reading the Side Lengths



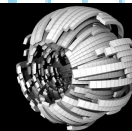
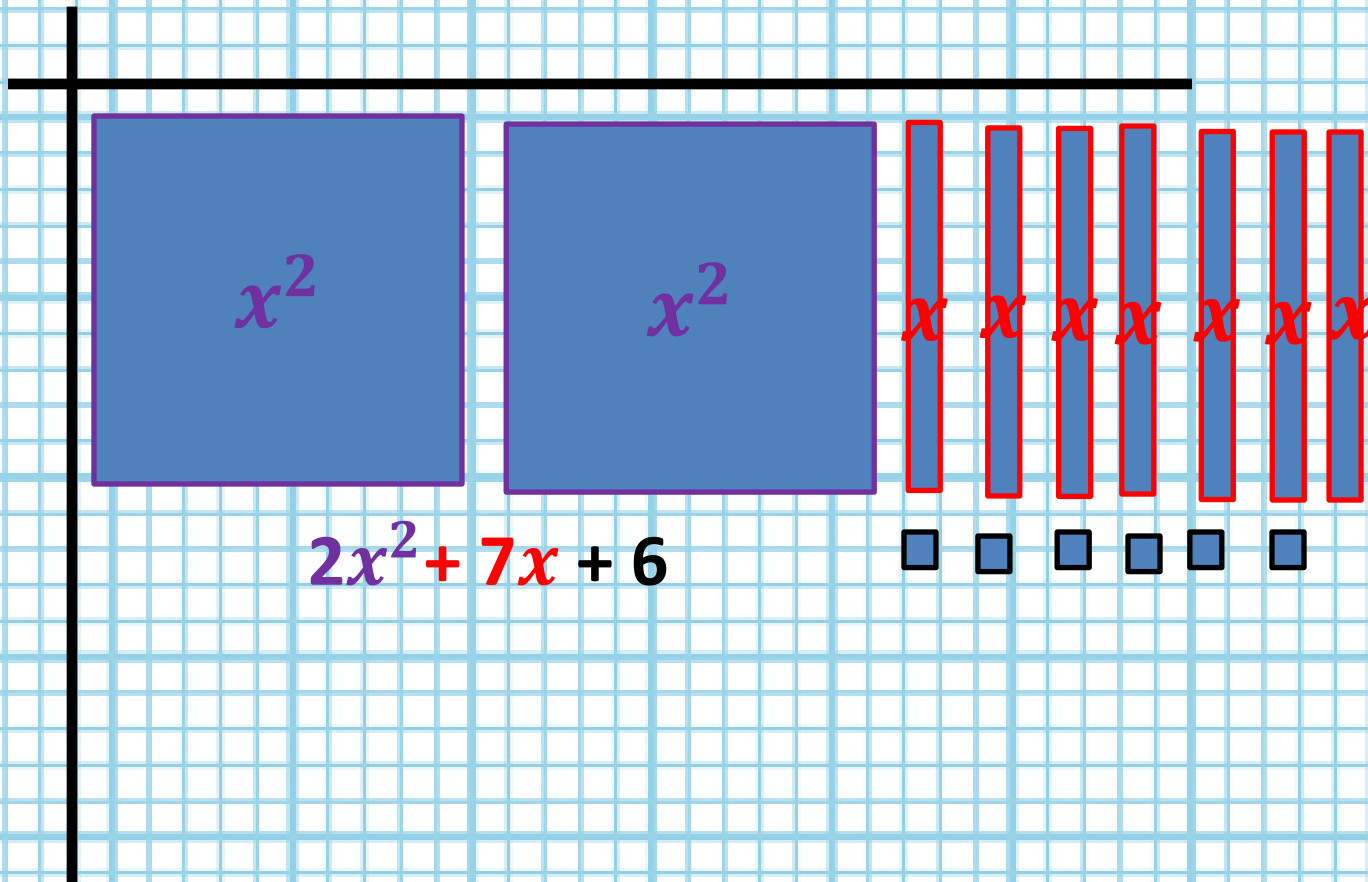
Students often struggle to see the length of the sides on the top and on the left. If they cannot see it there, then use the opposite sides. **Remind them it is the length of the side and not the area of the entire shape.**



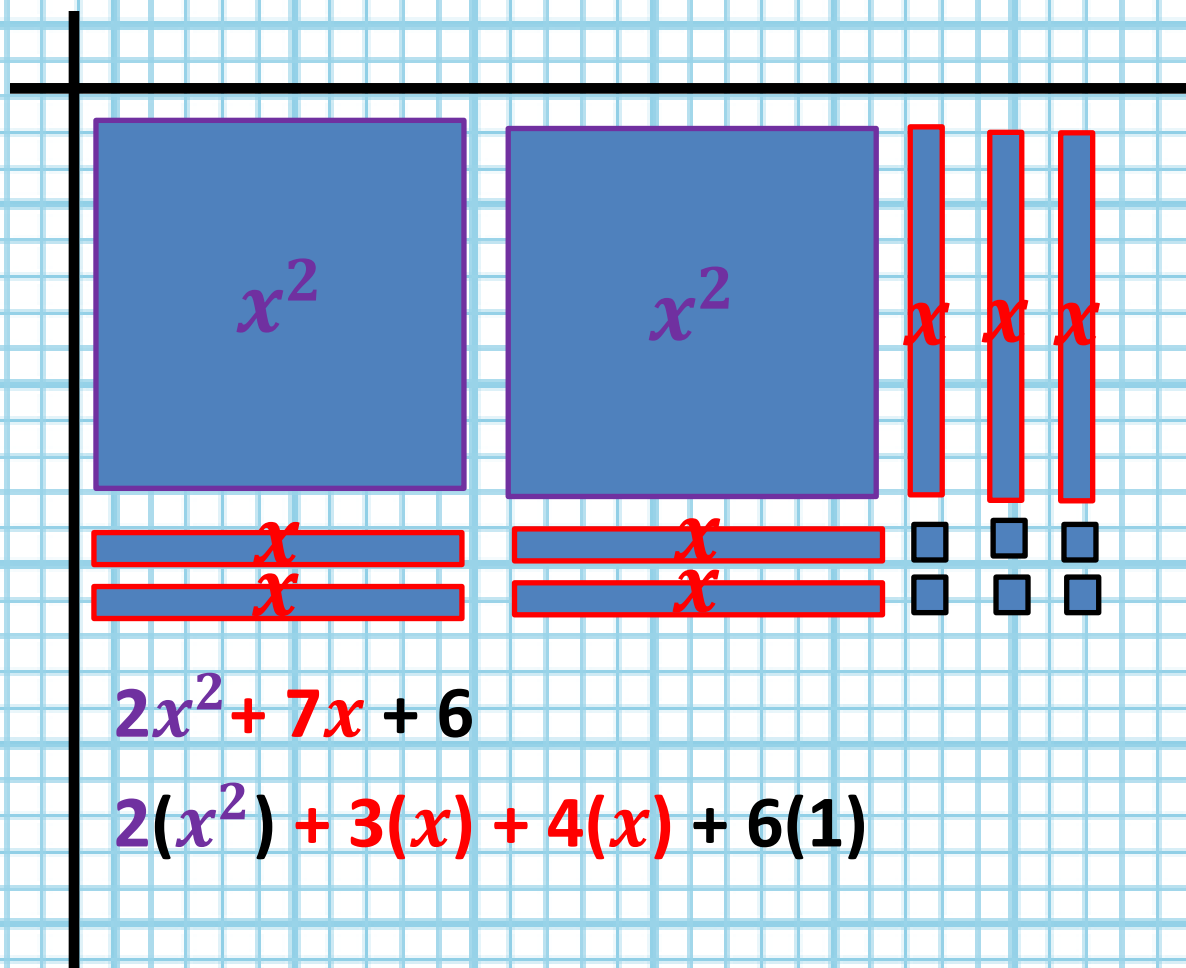
With a Partner: Factor $2x^2 + 7x + 6$ using Algebra Tiles

Remember multiplying polynomials and factor $2x^2 + 7x + 6$

(put it in a rectangle and find the side lengths)

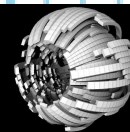


With a Partner: Factor $2x^2 + 7x + 6$, The Rectangle

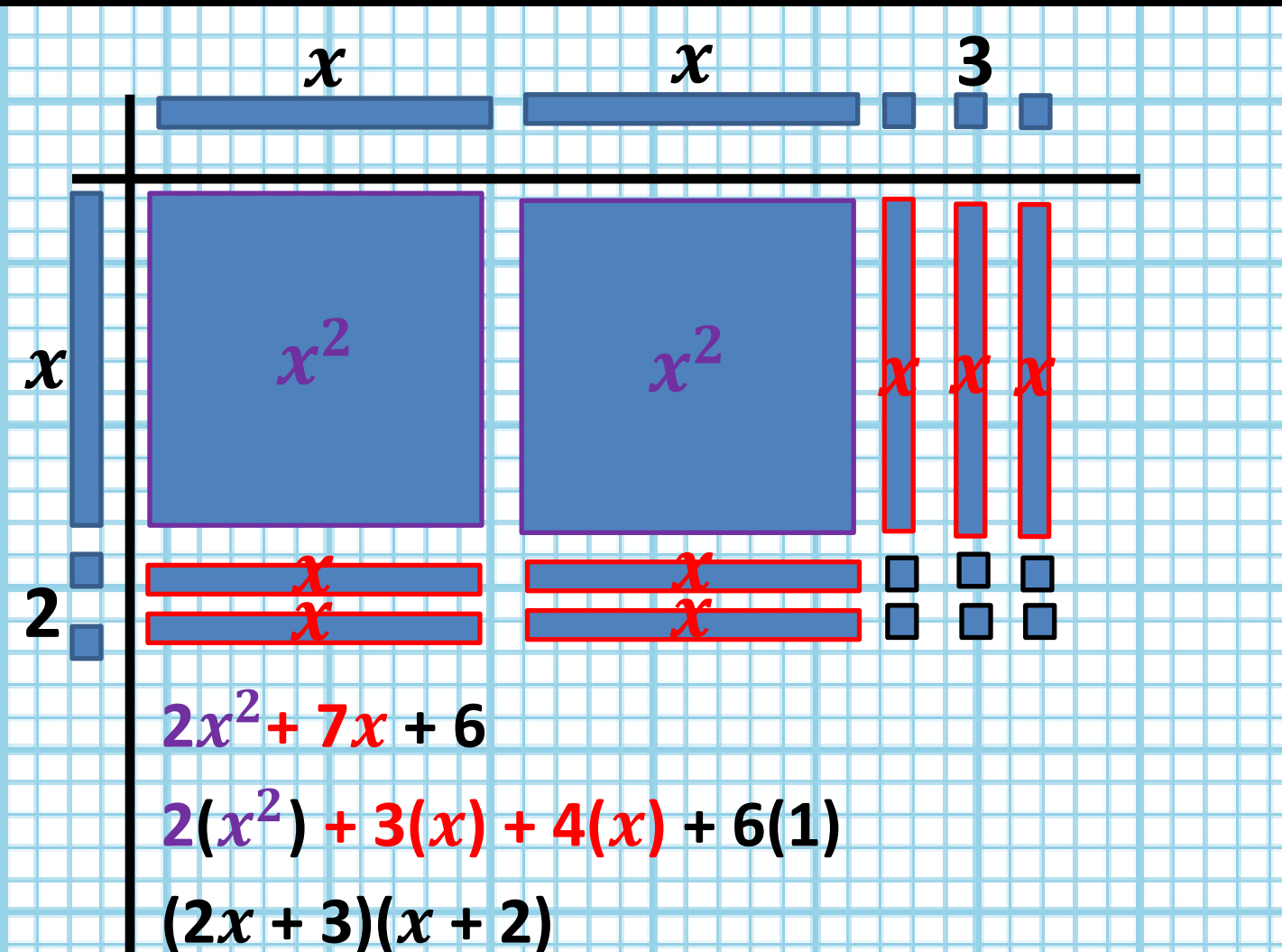


Emphasize that it has to multiply to make the quadratic and constant terms while adding correctly to make the linear terms. This is the X in X-box coming next.

Read the lengths of each side of your rectangle and write in the factors on the frame on your whiteboard.

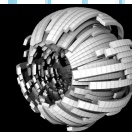


With a Partner: Factor $2x^2 + 7x + 6$, The Rectangle

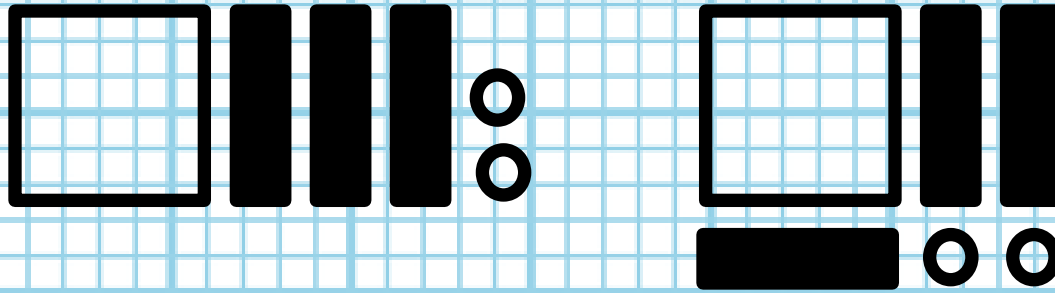


The length of the two sides are your factors.

Emphasize that both expressions, $2x^2 + 7x + 6$ and $(2x + 3)(x + 2)$, are equivalent, both representing the area taken up by the shapes. By the way, less pain right?

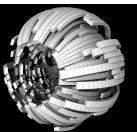


Next Level of Abstraction: Draw the Shapes



Open terms
are positive
and closed
terms are
negative.

Referring to
them as white
being positive
and black
being
negative is
not a good
idea.

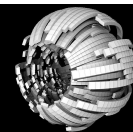


Factoring Polynomials: Four Column Notes

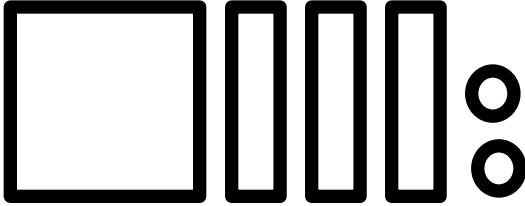
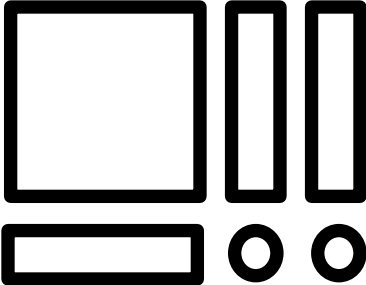

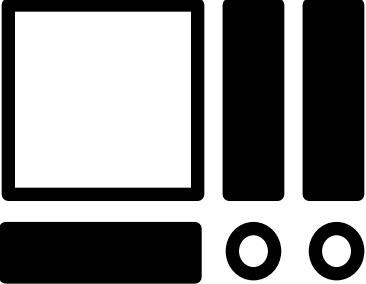
Fold Your Paper into Four Columns

<u>Unfactored Expression</u>	<u>Pictorial Expression</u>	<u>Pictorial Factored</u>	<u>Factored Expression</u>

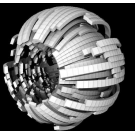
**Isn't this just the multiplication one backwards?
Exactly!**



Factoring Polynomials: Four Column Notes

<u>Unfactored Expression</u>	<u>Pictorial Expression</u>	<u>Pictorial Factored</u>	<u>Factored Expression</u>
$x^2 + 3x + 2$			$(x + 2)(x + 1)$
$x^2 - 3x + 2$			$(x - 2)(x - 1)$

I included a negative term here as an example if you choose to go there before higher levels of abstraction.



Move Fluidly between Representations

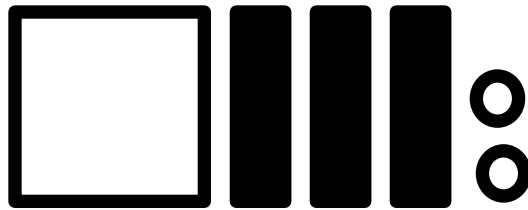
Unfactored
Expression

Pictorial
Expression

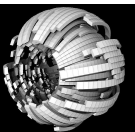
Pictorial
Factored

Factored
Expression

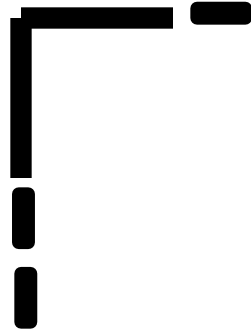
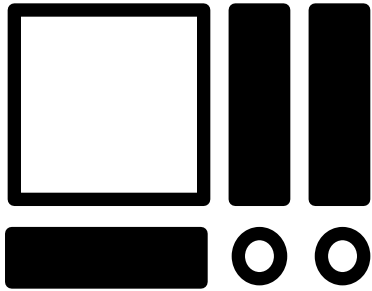
$$x^2 + 3x + 2$$



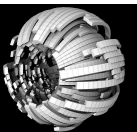
Once they have practiced, you can give them a single expression and ask for the others.



Move Fluidly between Representations

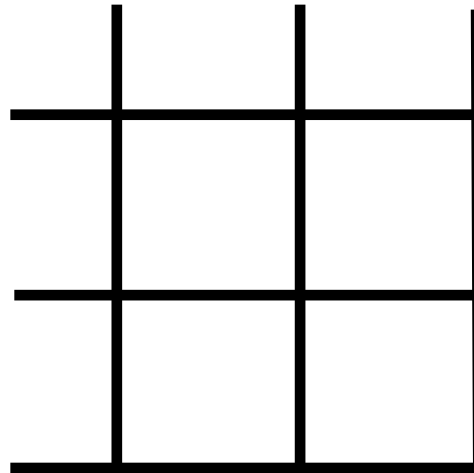
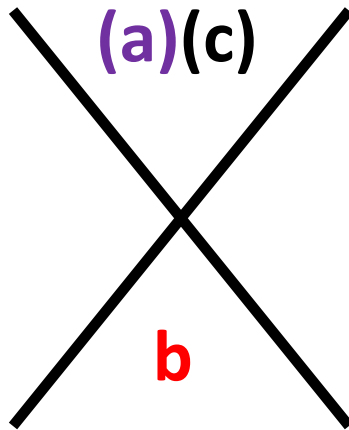
<u>Unfactored Expression</u>	<u>Pictorial Expression</u>	<u>Pictorial Factored</u>	<u>Factored Expression</u>
			$(x + 2)(x + 1)$
			

The top one is the most difficult so I included a method of drawing the length of each side before completing the picture.

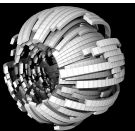


Next Level of Abstraction: X Box

$$ax^2 + bx + c$$

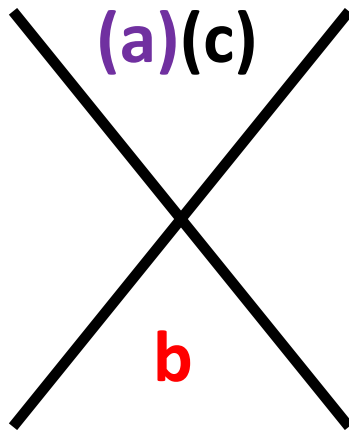


Sorry, no
gaming
today...



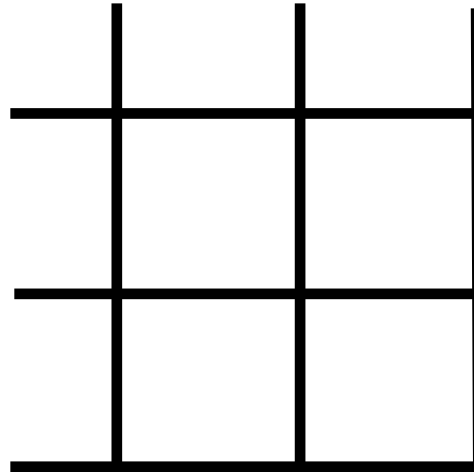
It is Not a Box; It is a Representation of Area

Linear Terms Must Multiply to make the Quadratic



Linear Terms Must Add (Combine) to Make the Final Linear Term

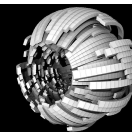
$$ax^2 + bx + c$$



The box method is a level of abstraction of the area method. Taught without the algebra tiles it means no more than numbers in a box like a multiplication table. Algebra tiles should be done first so when students use the Xbox they visualize the algebra tiles.

Terms within the boxes represent the area of those boxes while the factors on the edge of the box represent the lengths of the corresponding sides.

You can use algebra tiles for negatives; however, in the interest of time now is good time to bring them in to the instruction.



Let's Play X-Box: Insert Quadratic & Constant

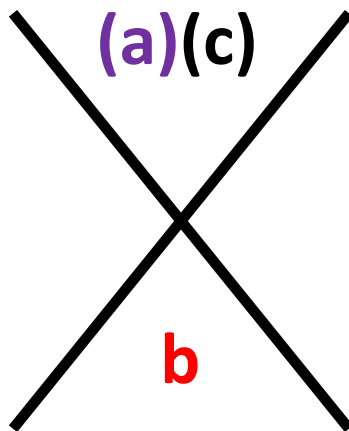
I extend the lines on the "box" to resemble length notation and make any negative signs more obvious, different than subtraction.

We are not really playing Xbox. We know the areas and are trying to find the factors, or the side lengths.

Remember the quadratic term goes in the top left and the constant goes in the bottom right.

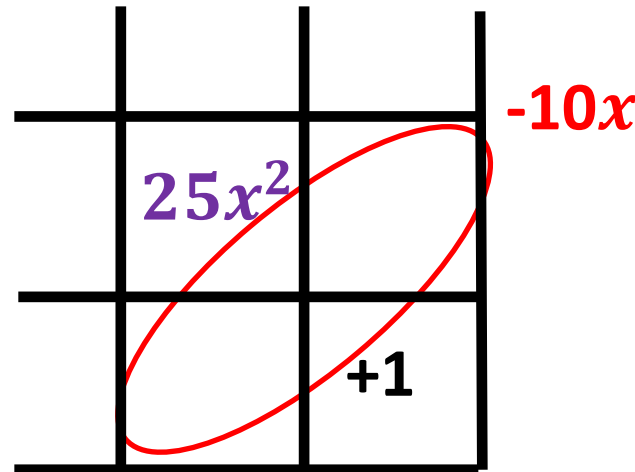
The question is how do I get $-10x$? There are an infinite number of ways to do so, which is why we have the X.

$ax^2 + bx + c$
Linear Terms Must
Multiply to make the
Quadratic

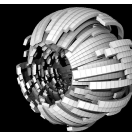


Linear Terms Must Add
(Combine) to Make the
Final Linear Term

$$25x^2 - 10x + 1$$



Now is also a good time to make the
coefficients larger.



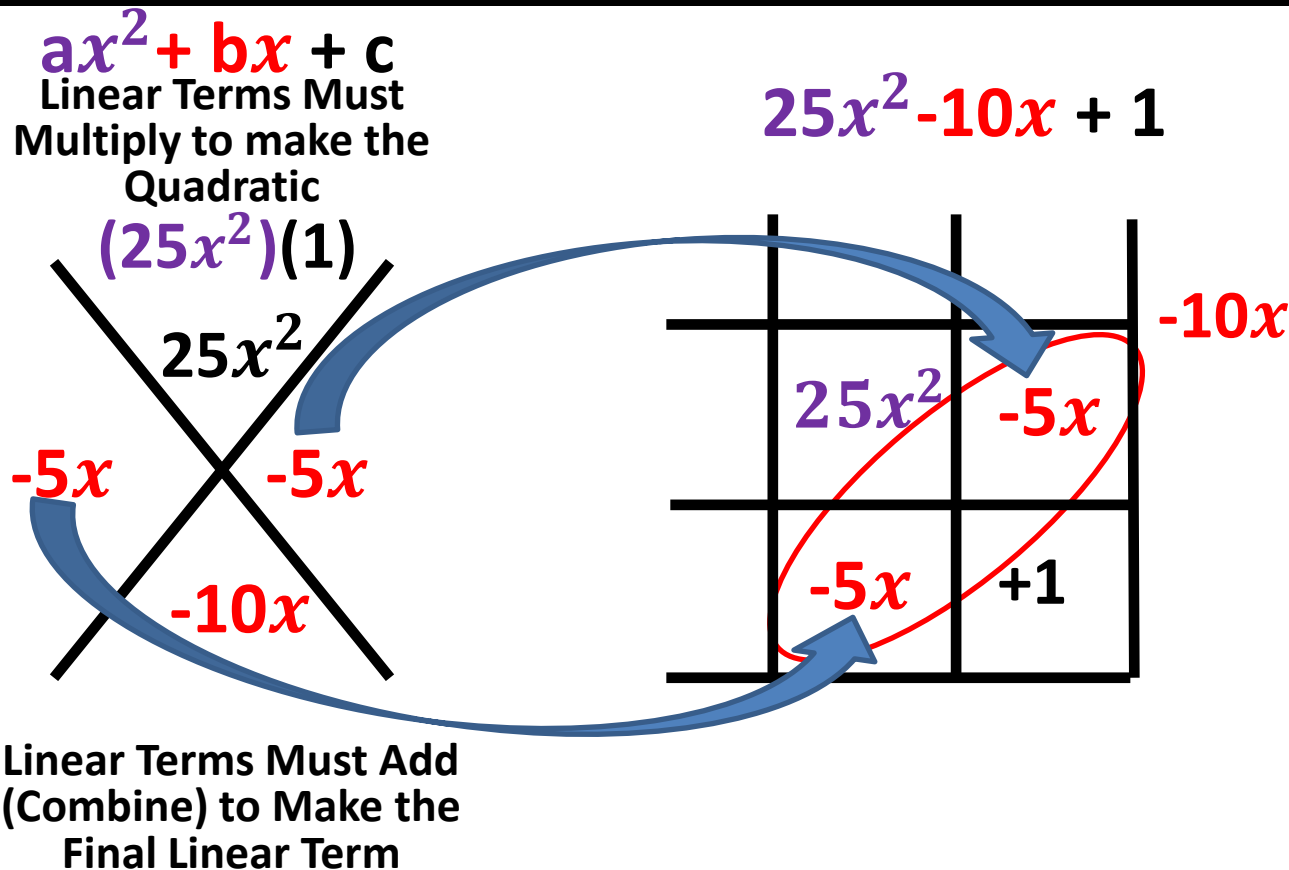
Let's Play X-Box: Use X to Find the Correct Linear Terms

What two numbers multiply to 25 and add (combine) to make -10?

You can use the coefficients only on the X as another level of abstraction.

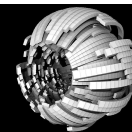
By the way, this is a Perfect Square Trinomial.

Difference of Two Squares, Perfect Square Trinomials, and Completing the Square should be done with algebra tiles first to allow students to discover the patterns or learn the rules. (SMP #8)



These X's are awesome for practicing integer operations in seventh grade and preparing students for Algebra 1.

See X Marks the Spot by Brad Fulton at tttpress.com



Let's Play X-Box: Find the Greatest Common Factor

$$ax^2 + bx + c$$

Linear Terms Must
Multiply to make the
Quadratic

$(25x^2)(1)$
 $25x^2$
 $-5x$ $-5x$
 $-10x$

Linear Terms Must Add
(Combine) to Make the
Final Linear Term

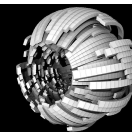
$$25x^2 - 10x + 1$$

$5x$	$25x^2$	$-5x$
	$-5x$	$+1$

Be sure to
factor out the GCF
before doing the
Xbox and the
quadratic term
should be positive.

If $a=1$, then you
don't need the
box; however, you
want students to
discover this to
help them
remember when to
do it and when not
to.

Find the
Greatest Common
Factor once for any
row or column.
Only take a
negative if it is
common to both .



Let's Play X-Box: Division or GCF Again

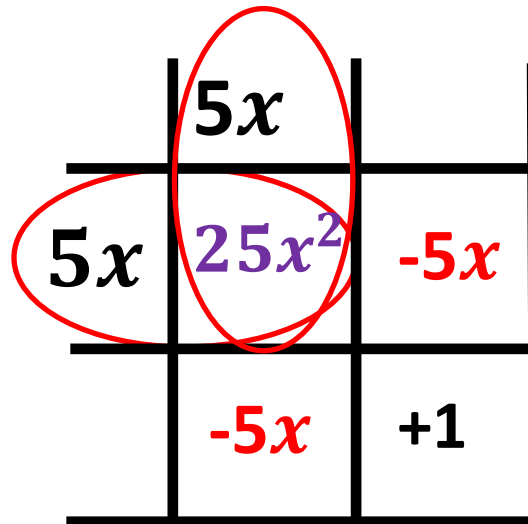
$$ax^2 + bx + c$$

Linear Terms Must
Multiply to make the
Quadratic

$$\begin{array}{cc} (25x^2)(1) & \\ \swarrow & \searrow \\ 25x^2 & \\ \swarrow & \searrow \\ -5x & -5x \\ \swarrow & \searrow \\ -10x & \end{array}$$

Linear Terms Must Add
(Combine) to Make the
Final Linear Term

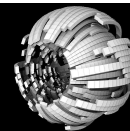
$$25x^2 - 10x + 1$$



Once you know a single length, then you can use it to find the rest or just keep using GCF.

Area is length times width so you can use this to find the rest of the side lengths.

What times $5x$ is $25x^2$?



Let's Play X-Box: Division or GCF Again

$$ax^2 + bx + c$$

Linear Terms Must
Multiply to make the
Quadratic

~~$(25x^2)(1)$~~
 $25x^2$
 $-5x$ $-5x$
 $-10x$

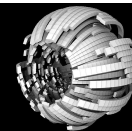
Linear Terms Must Add
(Combine) to Make the
Final Linear Term

$$25x^2 - 10x + 1$$

	$5x$	-1
$5x$	$25x^2$	$-5x$
	$-5x$	$+1$

What times $5x$
is $-5x$?

Remember to
point out you
are using the
area of the
rectangle and
one side to
find the
remaining
side.



Let's Play X-Box: Division or GCF Again

$$ax^2 + bx + c$$

Linear Terms Must
Multiply to make the
Quadratic

$$\begin{array}{cc} (25x^2)(1) & \\ & 25x^2 \\ -5x & -5x \\ & -10x \end{array}$$

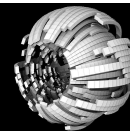
Linear Terms Must Add
(Combine) to Make the
Final Linear Term

$$25x^2 - 10x + 1$$

	$5x$	-1
$5x$	$25x^2$	$-5x$
-1	$-5x$	$+1$

What times $5x$
is $-5x$?

Of course, if
students
recognized
this as a
perfect square
trinomial,
then they
would only
have to do
one side...



Let's Play X-Box: Check It !

$$ax^2 + bx + c$$

Linear Terms Must
Multiply to make the
Quadratic

$$\begin{array}{c} (25x^2)(1) \\ 25x^2 \\ -5x \quad -5x \\ -10x \end{array}$$

Linear Terms Must Add
(Combine) to Make the
Final Linear Term

$$25x^2 - 10x + 1$$

	$5x$	-1
$5x$	$25x^2$	$-5x$
-1	$-5x$	$+1$



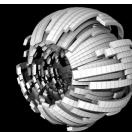
If it has been
done correctly,
then the final
area should
check out.

Does -1 times -1
equal $+1$?

By the way, the
base ten
equivalent, if
 $x=10$, would be
 $(49)(49)$

$$\begin{aligned} &(50-1)(50-1) \\ &2500-50-50+1 \\ &2401 \end{aligned}$$

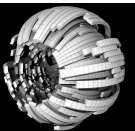
... a great way
to introduce
special
products!



Final Level of Abstraction: Just do it! (Guess and Check)

$$25x^2 - 10x + 1$$

$$(5x - 1)(5x - 1) \text{ or } (5x - 1)^2$$



What Materials Did I Use?

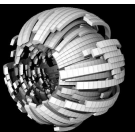
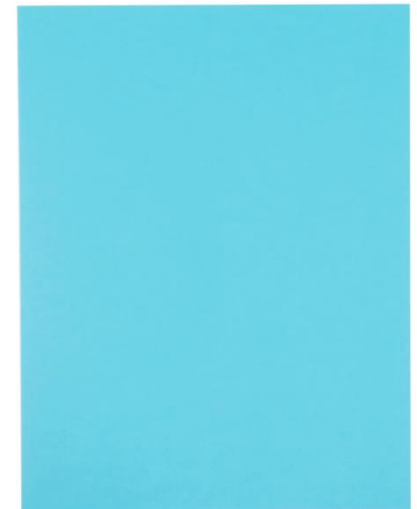
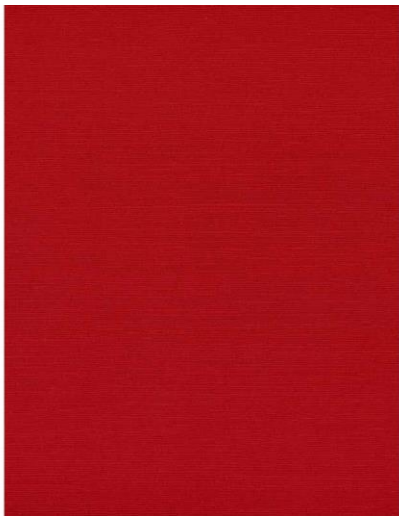
Two colors of cardstock

Slider bags

Folded blank paper

Cheapest manipulatives EVER!

And they do much more than this...

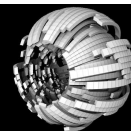


What about the Base Ten Blocks?

Some might argue against having the same manipulative as base ten and algebra tiles.

My counter argument is why keep two manipulatives when one will do?

More importantly, I *want* them to make connections between base ten and algebraic representations of any base to aid in understanding and retention.



Color and Algebra Tiles?

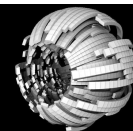
Store bought algebra tiles have “unknown” side lengths as they are trying to reinforce the that quality of the variable.

If you measure these tiles with the one, then you find they are usually between three and four. Does that mean it has a fractional side length?

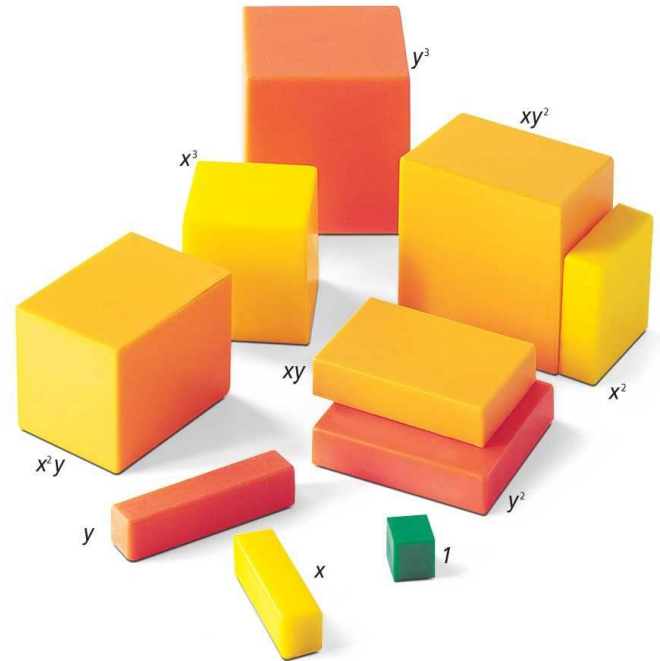
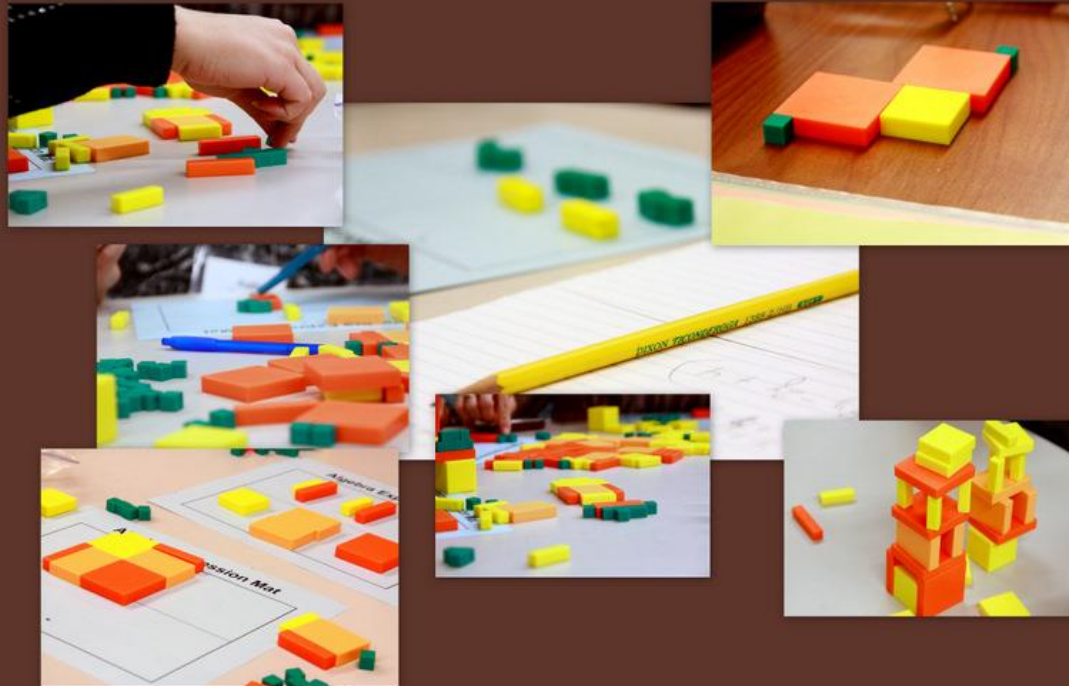
All models have inherent flaws; choose the flaws that reinforce the learning when you can.



I don't prefer the multiple color tiles. I prefer to use color to distinguish between positive and negative or different variables.

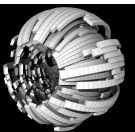


What about Y? Color & Algeblocks



AlgeBlocks allow for work in three dimensions, cubic expressions & equations, and work with two variables, x & y.

<http://www.hand2mind.com/item/algeblocks-manipulative-starter-set/9241> Expensive @ \$200 though.



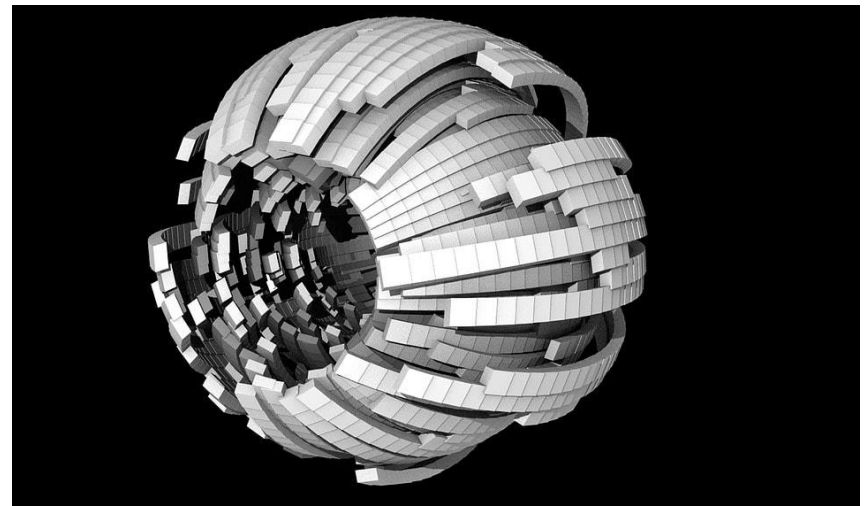
Presentation, Questions & Feedback

This presentation will be posted on my fledgling website www.meaningformemory.com

Email me at david.mattoon@gmail.com if you are interested in larger, precut consumable manipulatives.

Please leave [feedback & questions](#) on the site.

Using Base Ten to See & Understand Algebra



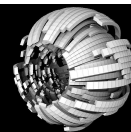
Using Algebra to See & Understand Base Ten

Fall for Math Feedback

Please go to the following url to give your honest feedback about this session. Your feedback helps the Conference Committee in selecting sessions for future conferences and gives feedback to presenters so they can learn from this experience.

Breakout #2 Feedback:

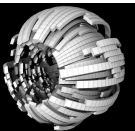
<https://tinyurl.com/y9p38l36>



What are our words worth?



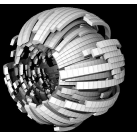
“I’ll give you a thousand words for it.”



A Picture is Worth a Thousand Words

~ Fred Barnard, 1927

The notion that a **complex idea** can be conveyed with just a **single still image** or that **an image of a subject conveys its meaning or essence more effectively** than a description does.



Ever get tired of repeating yourself ?

- **"Hearing something a hundred times isn't better than seeing it once"**
~ Chinese Expression
- **"The drawing shows me at a glance what would be spread over ten pages in a book."** ~ Ivan Turgenev, Russian Author
- **"A good sketch is better than a long speech"** ~ Napoleon Bonaparte, Emperor of the French

