All Students Shine with Bar Models from Subtraction to Systems

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Please try the problem.

BEAUTY & BRILLIANCE
CMC-S 2019
60TH ANNIVERSARY

Origin of the Equal Sign

Robert Recorde, 1557

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Mattoon's MEANING 4 MEMORY

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What about the equals sign?

Where did the equal sign come from?

What does it represent?

How does it demonstrate equivalence?

=

What about the equals sign?

$14x + 15 = 71$

Where did the equal sign come from?

What does it represent?

How does it demonstrate equivalence?

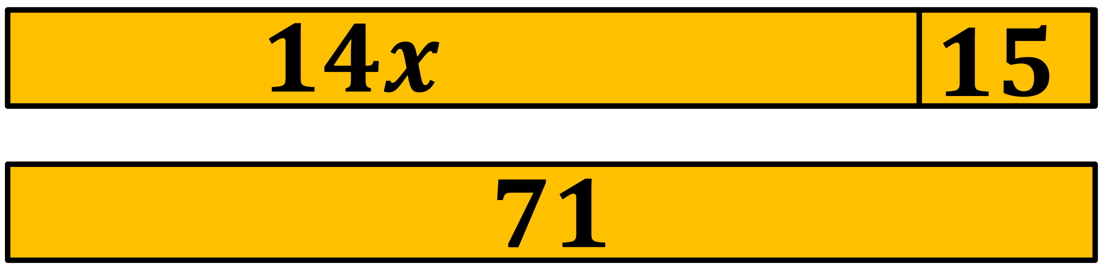
The etymology of the word "equal" is from the Latin word "æqualis" as meaning "uniform", "identical", or "equal", from aequus ("level", "even", or "just").

The "=" symbol that is now universally accepted in mathematics for equality was first recorded by Welsh mathematician Robert Recorde in The Whetstone of Witte (1557). The original form of the symbol was much wider than the present form. In his book Recorde explains his design of the "Gemowe lines" (meaning twin lines, from the Latin gemellus.

"And to avoid the tedious repetition of these words: is equal to: I will set as I do often in work use, a pair of parallels, or Gemowe lines of one length, thus: =, because no 2 things, can be more equal."

An equals sign is a tape diagram.

The first use of the equal sign by Robert Recorde was in the following problem: 14x + 15 = 71



Notice both sides of the equation are equivalent, which means both could be represented as two equivalent lengths or two equivalent areas.

<u>Twin lines of one length</u>, a pair of parallels, identical, even

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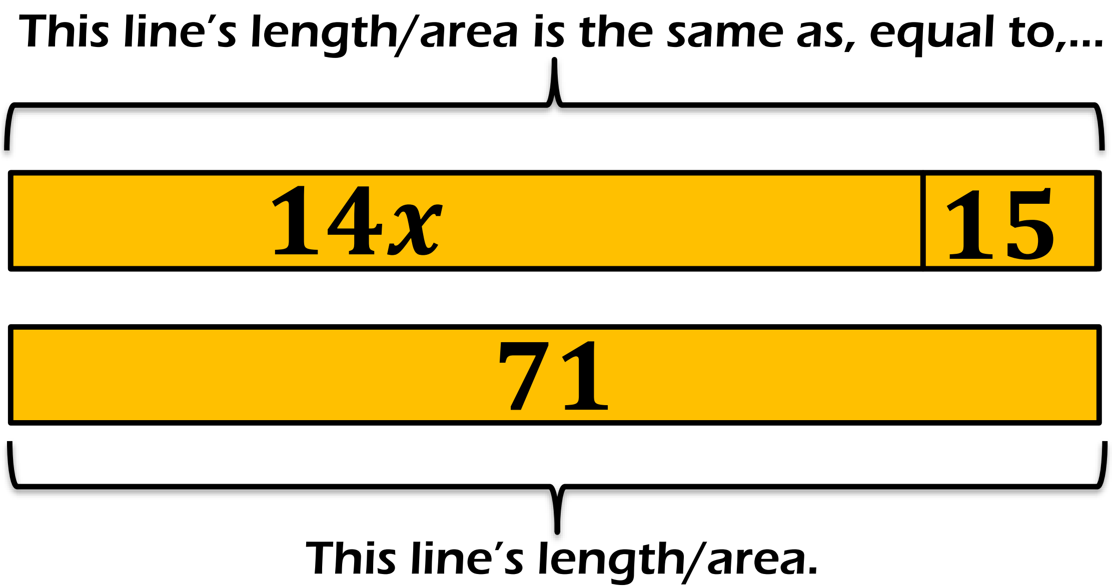
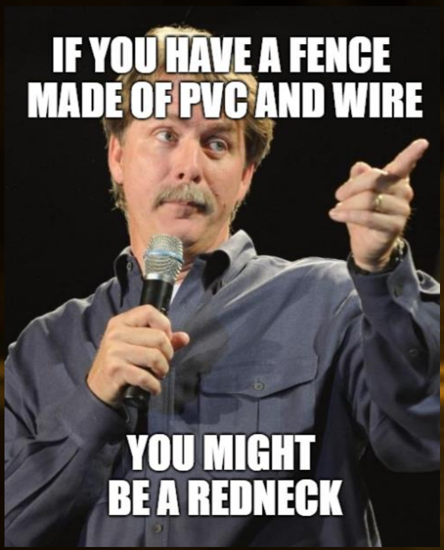


Notice both sides of the equation are equivalent, which means both could be represented as two equivalent lengths or two equivalent areas.

<u>Twin lines of one length</u>, a pair of parallels, identical, even

Real Life Word Problems

My Junky Fence



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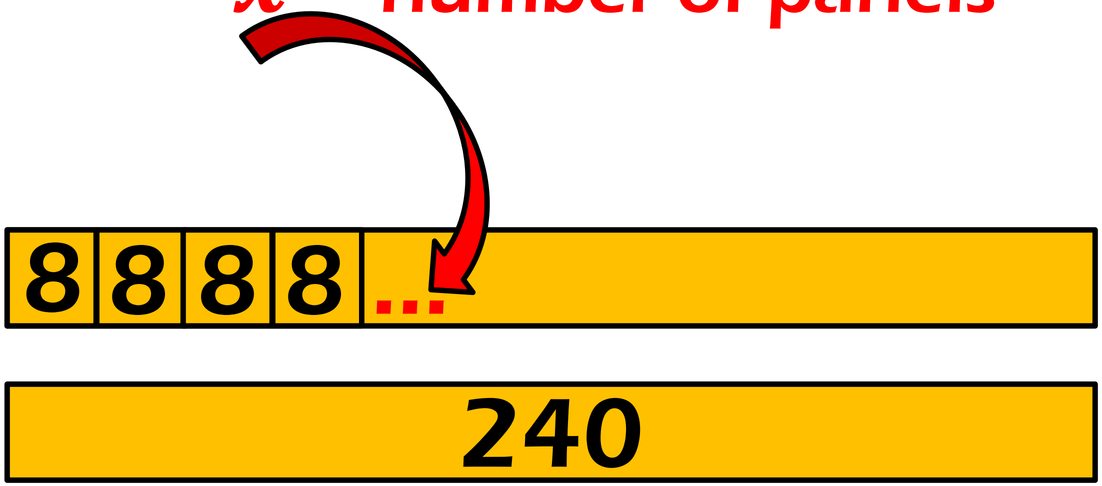
Real Life & Word Problems

Many real life problems and word problems, especially algebraic word problems, are grounded in equivalence.

How many eight foot long wood picket panels do I need to replace my junky fence that is 240 feet long?

x = number of panels



Real Life & Word Problems

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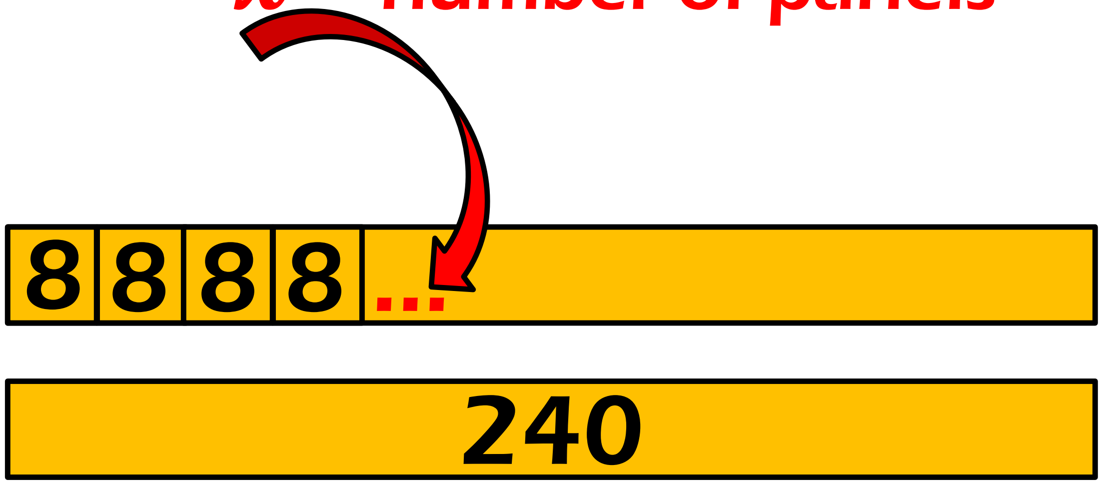

$$8x = 240$$

First Grade

Two Step Problems

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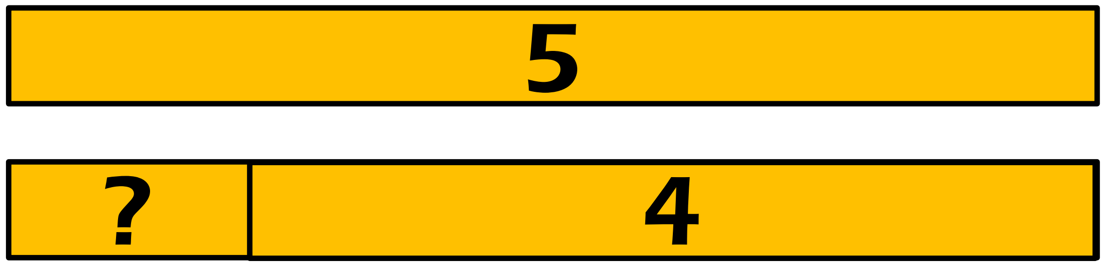
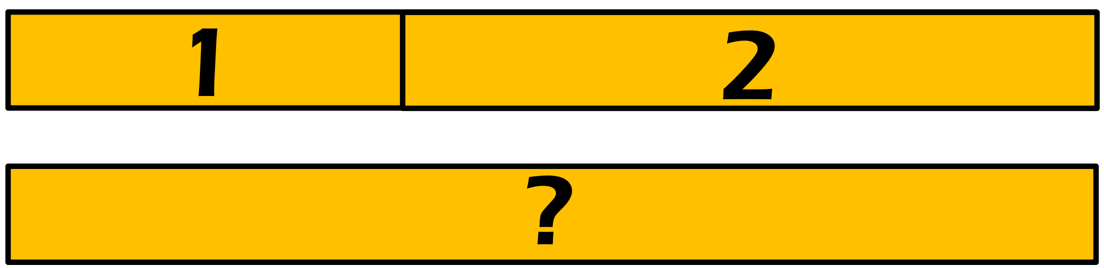
Applying Tape Diagrams to a First Grade, Two Step Problem

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

Step 1



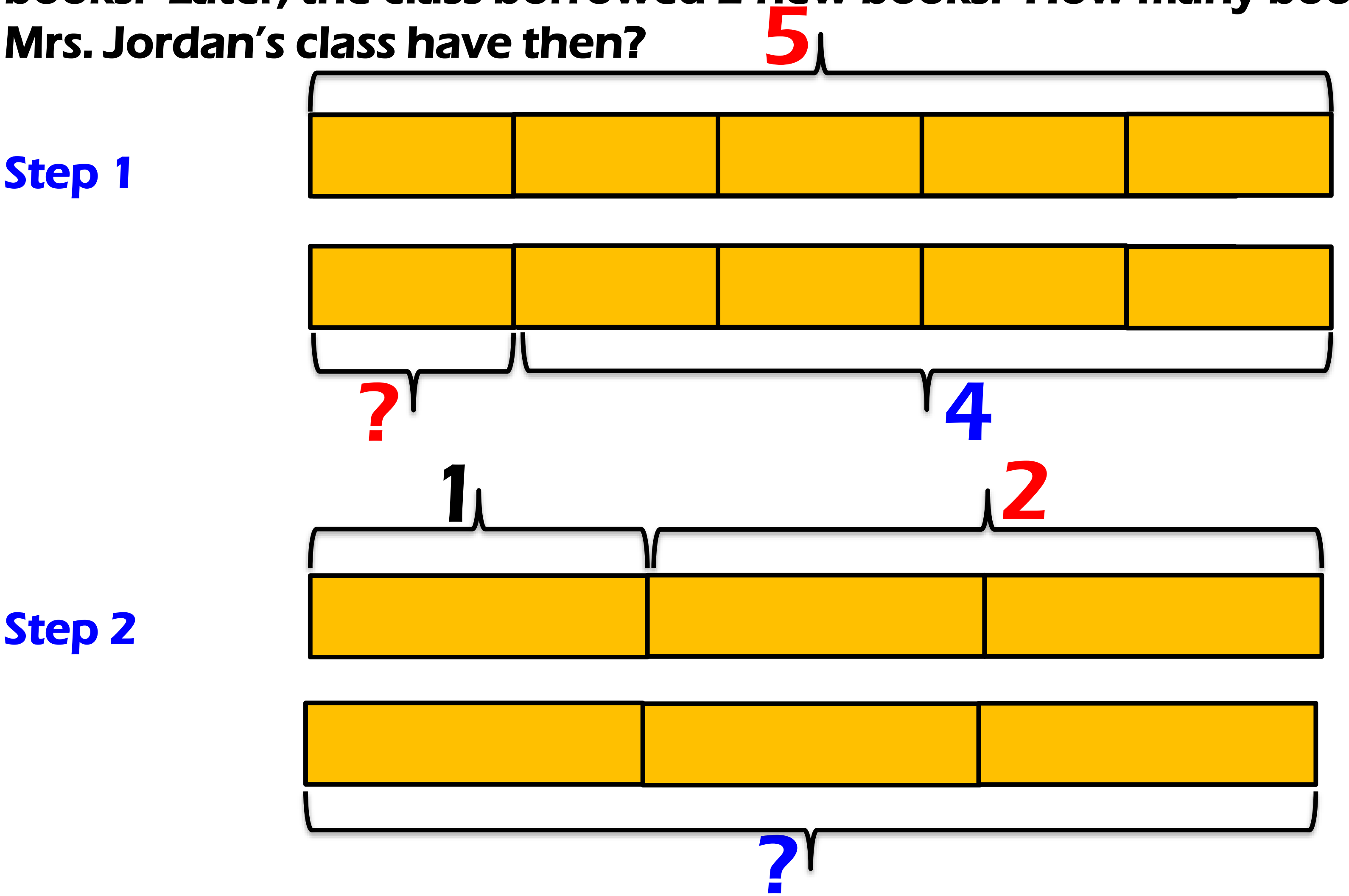
Step 2



This is an abstraction already. Why?

Initially, you may want to draw unit delineations.

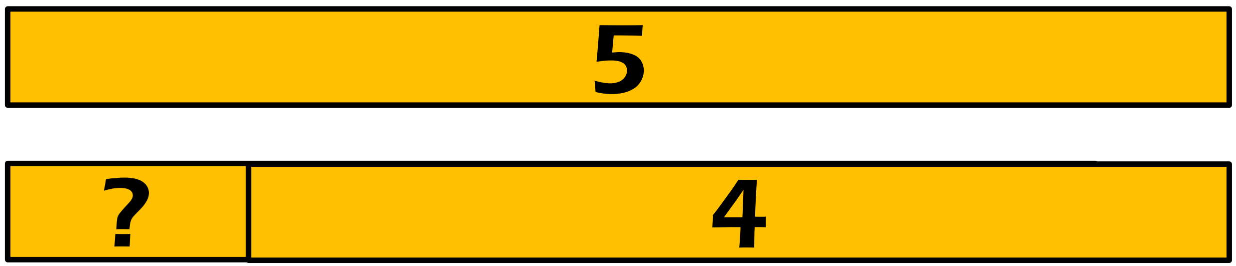
Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?



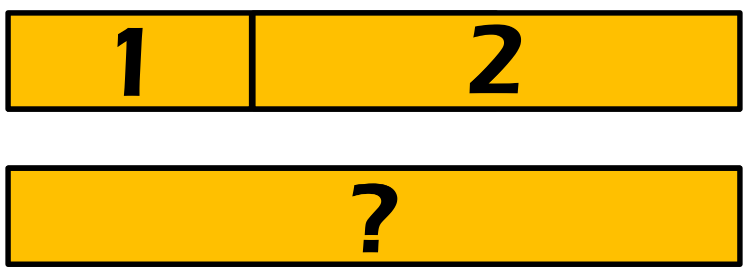
Scale

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

Step 1



Step 2



What about the concrete?

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

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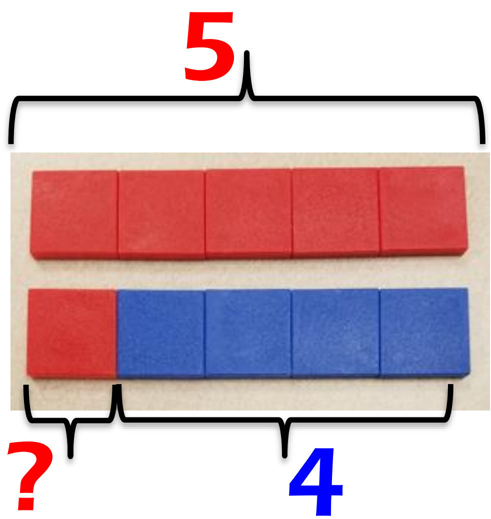
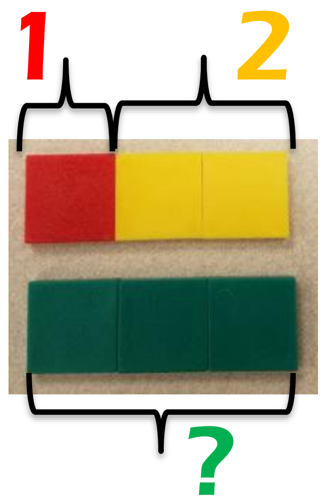
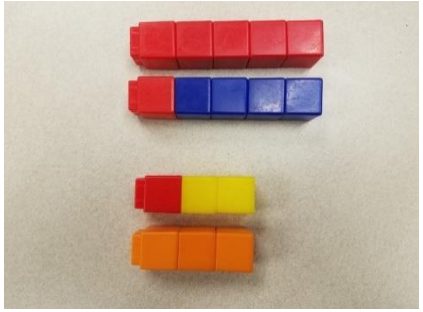


Step 2



Try your best to include the concrete stage.



What about the concrete?

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

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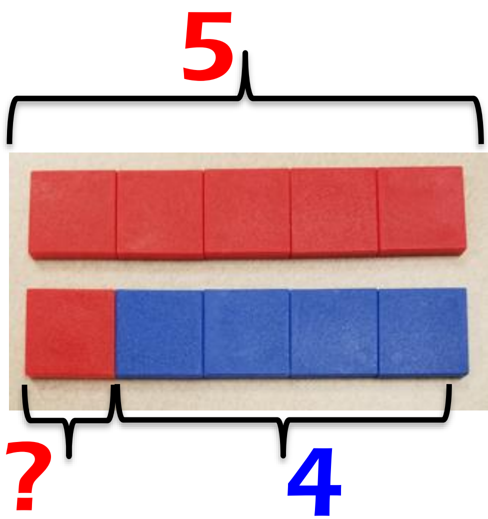
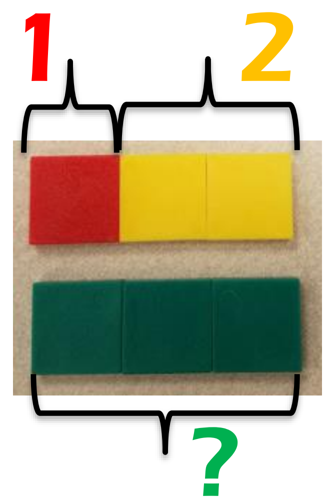
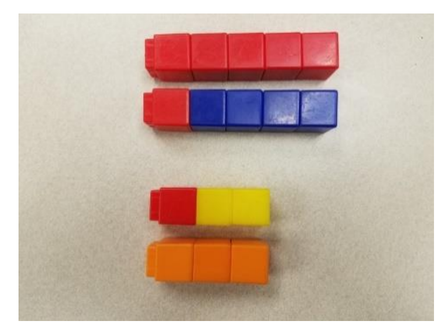


Step 2



The tape diagram models everything that happened in the problem; think of it as a history.



Concreteness Fading

Levels of Abstraction

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Levels of Abstraction = Concreteness Fading
Concrete, Representational, Abstract

During the first half of my teaching career, I would spend what seemed to be the first half of a math lesson teaching a new math concept by **sharing definitions, formulas, steps and procedures.**

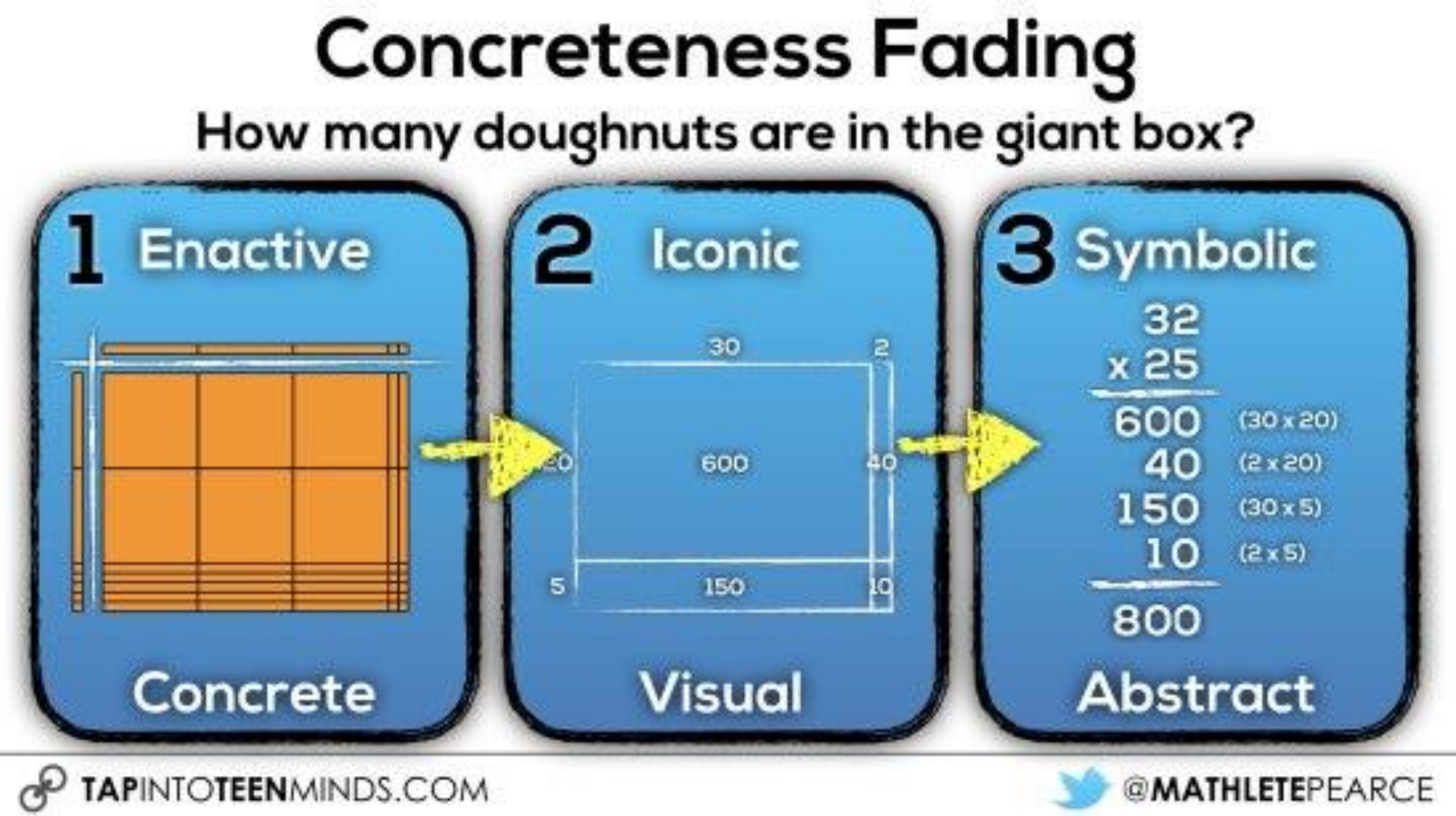
To make things more challenging for my students, **I would simultaneously introduce the symbolic notation** used to represent those ideas. Then, I would spend the remainder of the lesson **attempting** to help my students make sense of these very **new** and often **abstract ideas**.

By the end of the lesson, I could help many students build an understanding, but **there was always a group I felt who I would leave behind.** Like many other teachers, I was just **teaching in a very similar way to that how I was taught**. **I knew no different.**

However, if we consider that **new learning requires the linking of new information with information they already know and understand**, we should be intentionally planning our lessons with this in mind. A great place to start new learning is through the use of a **meaningful context** and **utilizing concrete manipulatives** that students can touch and feel.

When we teach in this way, we **minimize the level of abstraction** so students can **focus their working memory on the new idea** being introduced in a meaningful way.

- Kyle Pierce, Tap into Teen Minds, https://tapintoteenminds.com/concreteness-fading/



Jerome Bruner (1966) proposed three modes of representation:

Enactive representation (action-based)

Iconic representation (image-based)

Symbolic representation (language-based)

Concreteness Fading via CRA

CONCRETE

1

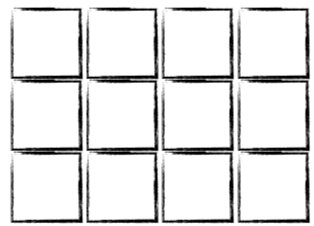
ACTUAL DONUTS

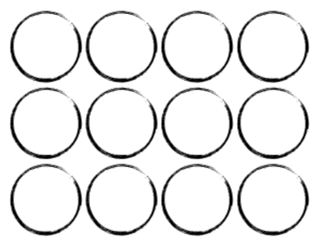
-or-

MANIPULATIVES

VISUAL

2





DRAWINGS & DIAGRAMS

ABSTRACT

3

$3 \times 4 = 12$

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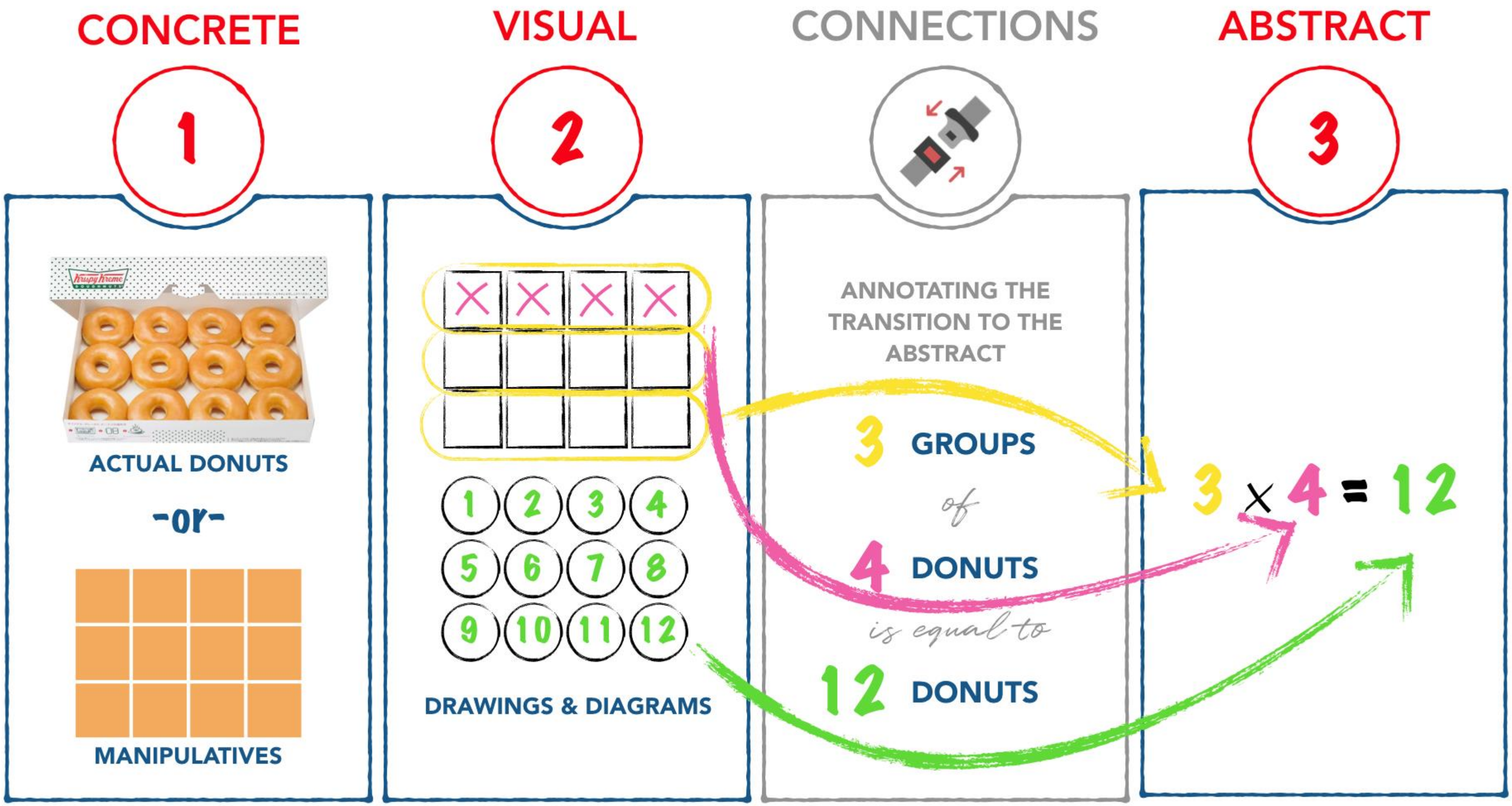
Concreteness Fading via CRA with CONNECTIONS!



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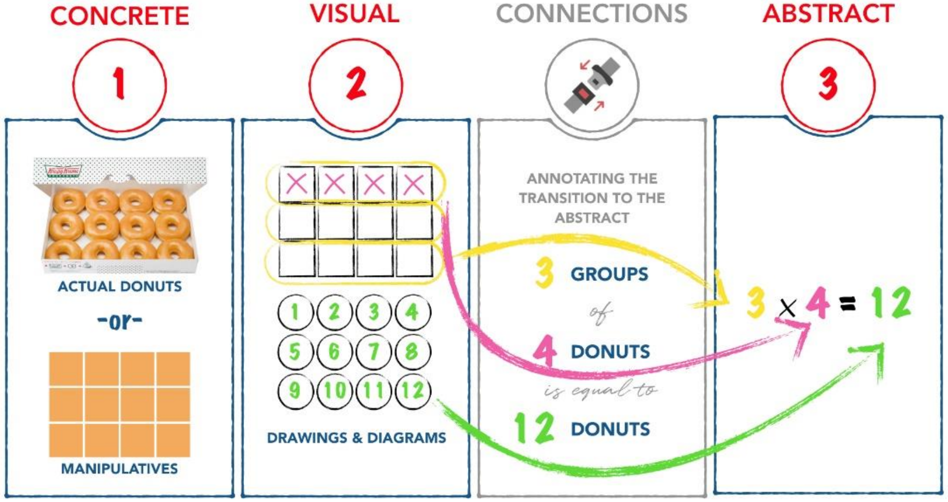
Gradual Release

John SanGiovanni
Howard County (MD)
Public Schools

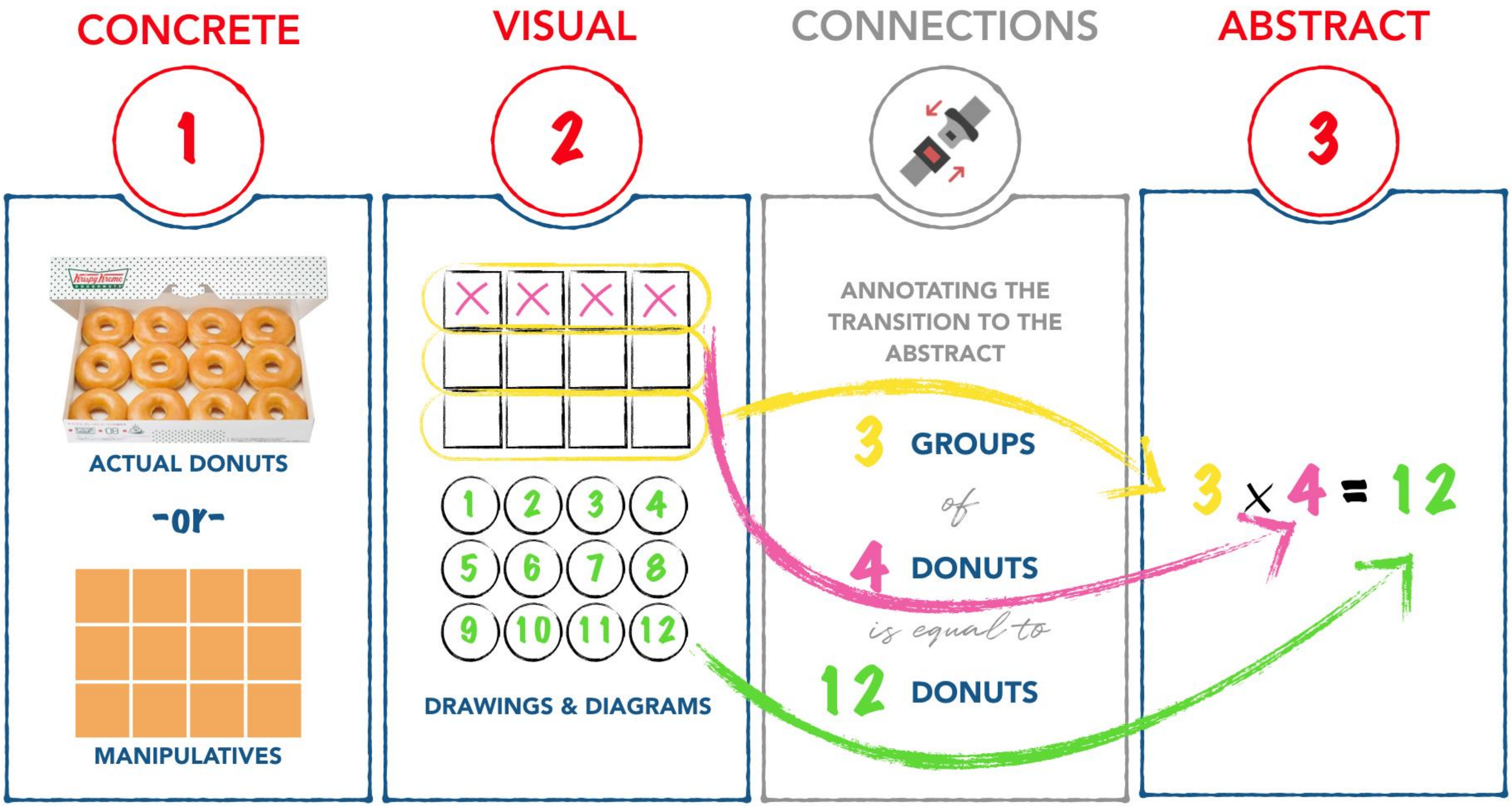
Rather than a gradual release from
I Do, We Do, You Do;

think of a gradual release from the
Concrete to the Abstract.



Concreteness Fading via CRA with CONNECTIONS!

Let's start again with this in mind.



First Grade

Two Step Problem with CRA

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Mattoon's MEANING 4 MEMORY

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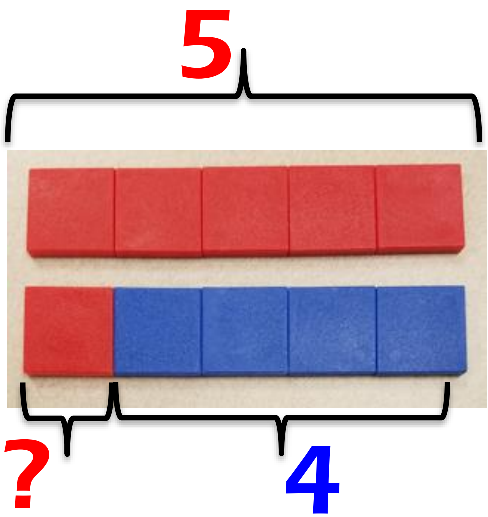
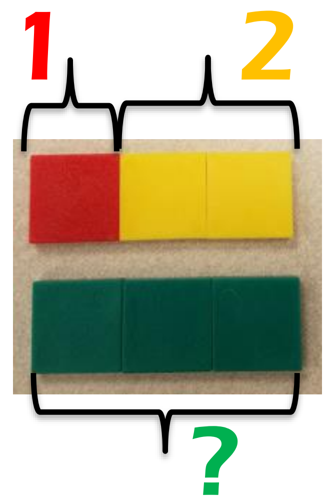
Concrete Stage (Manipulatives)

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

Step 1



Step 2



The students' manipulation of the tiles acts as a formative assessment and is an avenue into their thinking allowing for true scaffolding to take place.

Representational Stage (to scale)

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

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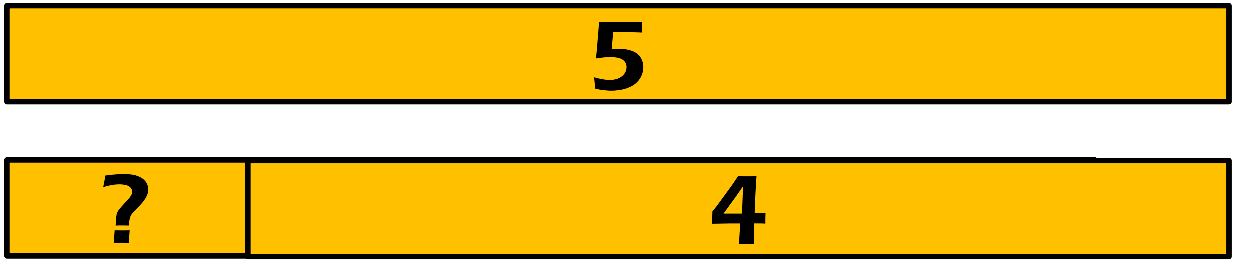
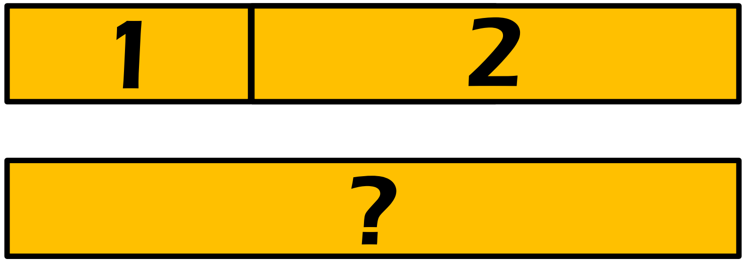


Step 2

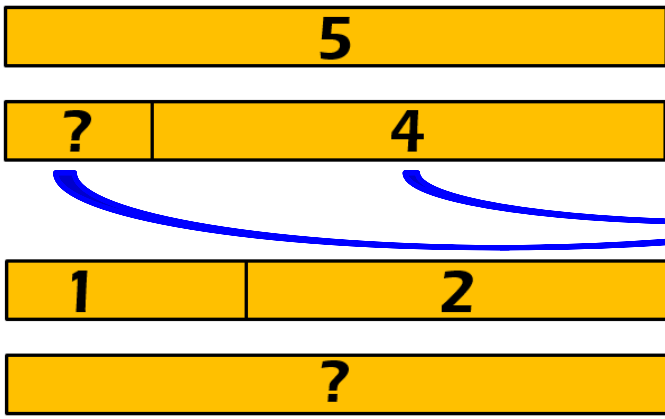


These boxes, tape diagrams, are to scale.

Connecting to the Abstract Stage

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

Step 1



$5 = 1 + 4$ or $5 - 4 = 1$

Step 2

$1 + 2 = 3$

1. Draw an equal sign
2. On one side, draw the values represented on one line.
3. On the other side, draw the values represented by the other line.

They are the same length!

Notice the directions do not specify which line to put where.

$$5 = 1 + 4$$

is just as valid as

$$1 + 4 = 5.$$

This helps in seeing the equal sign as equivalence rather than an operator. I have done it both ways above.

Final Level of Abstraction: Full Symbolic

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

Step 1 $5 - 4 = 1$

Avoid

$5 - 4 = 1 + 2 = 3$

Step 2 $1 + 2 = 3$

Why?

Final Level of Abstraction: Full Symbolic

Mrs. Jordan's class borrowed 5 books from the library. They returned 4 books. Later, the class borrowed 2 new books. How many books did Mrs. Jordan's class have then?

Step 1 $5 - 4 = 1$

Step 2 $1 + 2 = 3$

Avoid

$$5 - 4 = 1 + 2 = 3$$

as

$$5 - 4 \neq 1 + 2$$

and

$$5 - 4 \neq 3$$

Second Grade

Two Step Problem with CRA

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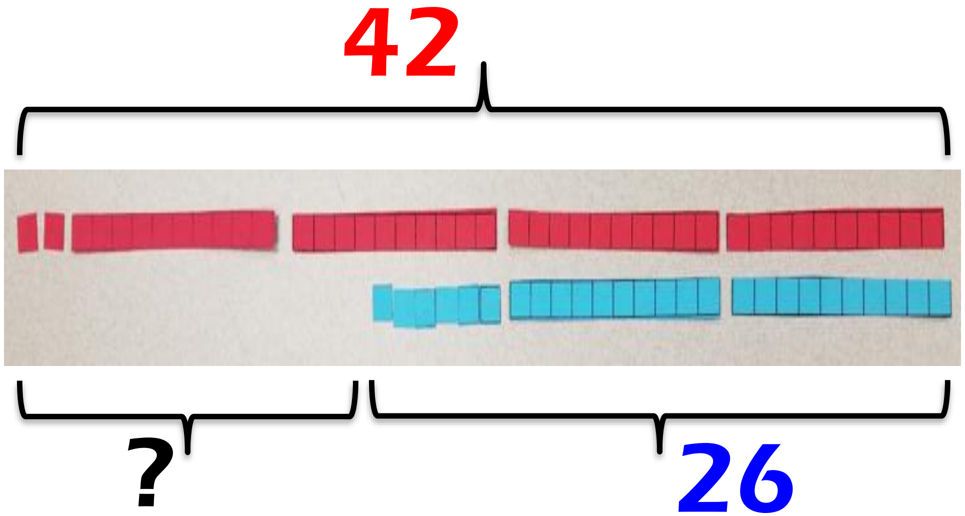
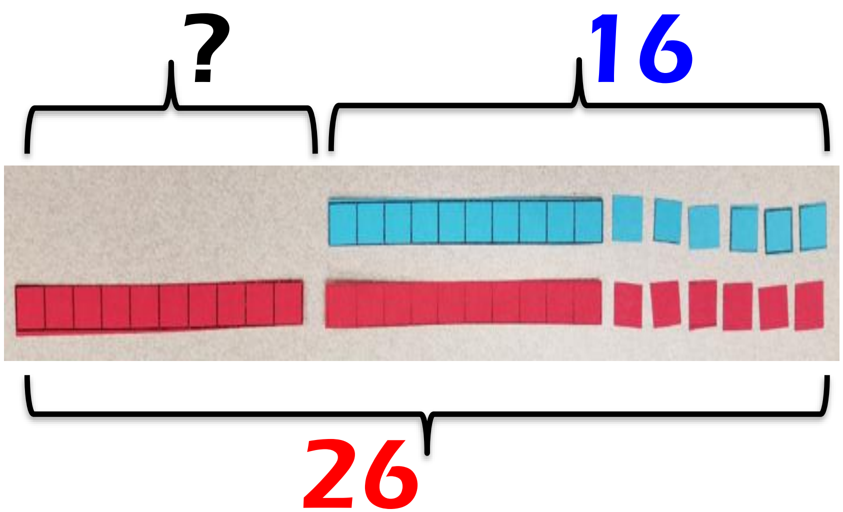
Concrete Stage (Manipulatives)

Astro works 42 hours at both the Black Hole and the Galaxy Grill. He works 26 hours at the Black Hole. How many fewer hours does Astro work at the Galaxy Grill?

Step 1



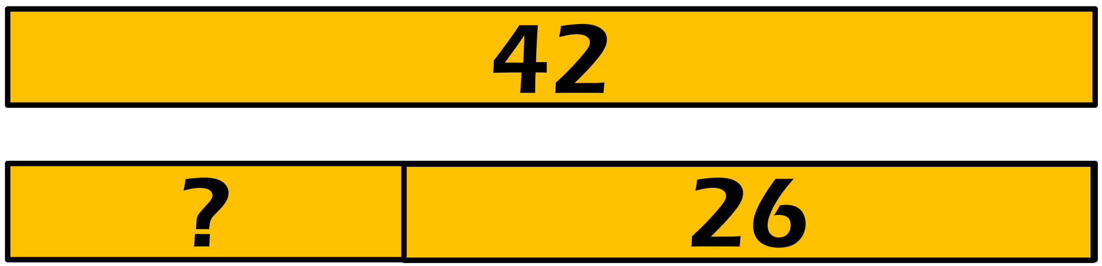
Step 2



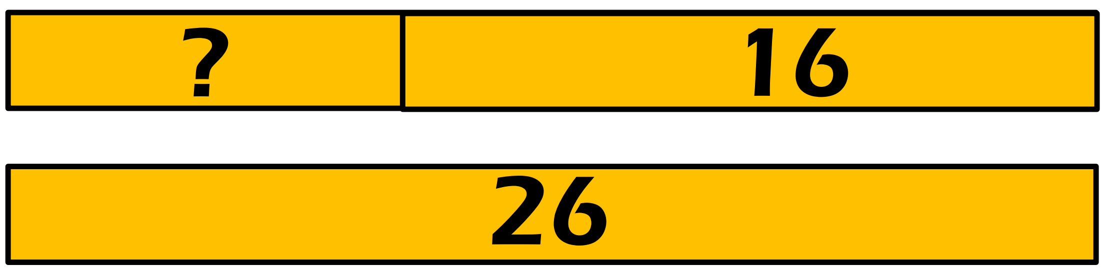
Representational Stage (not to scale)

Astro works 42 hours at both the Black Hole and the Galaxy Grill. He works 26 hours at the Black Hole. How many fewer hours does Astro work at the Galaxy Grill?

Step 1



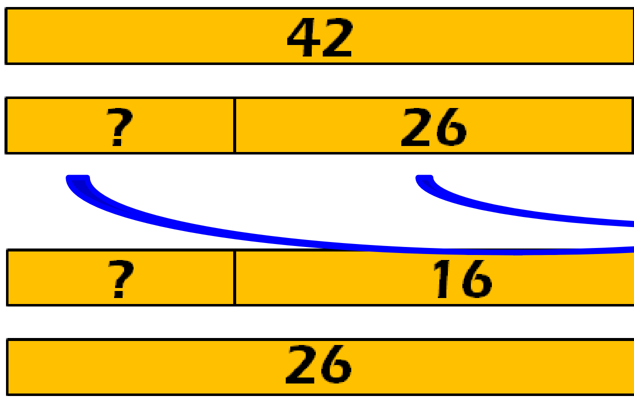
Step 2



Connecting to the Abstract Stage

Astro works 42 hours at both the Black Hole and the Galaxy Grill. He works 26 hours at the Black Hole. How many fewer hours does Astro work at the Galaxy Grill?

Step 1



$42 = 16 + 26$ or $42 - 26 = 16$

Step 2

$10 + 16 = 26$ or $26 - 16 = 10$

1. **Draw an equal sign**
2. **On one side, draw the values represented on one line.**
3. **On the other side, draw the values represented by the other line.**

They are the same length!

Notice the directions do not specify which line to put where.

$$42 = 16 + 26$$

$$\textit{is just as valid as}$$

$$16 + 26 = 42$$

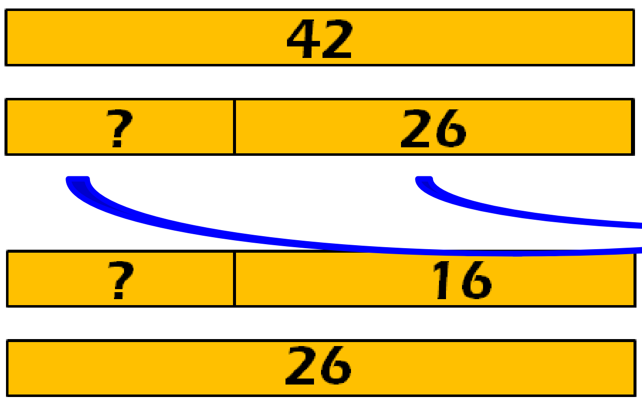
This helps in seeing the equal sign as equivalence rather than an operator. I have done it both ways above.

Connecting to the Abstract Stage with a Variable

Astro works 42 hours at both the Black Hole and the Galaxy Grill. He works 26 hours at the Black Hole. How many fewer hours does Astro work at the Galaxy Grill?

Step 1

Step 2



$42 = g + 26$ or $42 - 26 = g$

$h + 16 = 26$ or $26 - 16 = h$

1. Draw an equal sign
2. On one side, draw the values represented on one line.
3. On the other side, draw the values represented by the other line.

They are the same length!

You could use a single variable and redefine it, but for clarity I have defined two variables as

g= the # of hours worked at Galaxy Grill
h = the fewer # of hours

Final Level of Abstraction: Full Symbolic

Astro works 42 hours at both the Black Hole and the Galaxy Grill. He works 26 hours at the Black Hole. How many fewer hours does Astro work at the Galaxy Grill?

Step 1 $42 - 26 = 16$

Step 2 $26 - 16 = 10$

Third Grade

Two Problems with CRA

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Mattoon's MEANING 4 MEMORY

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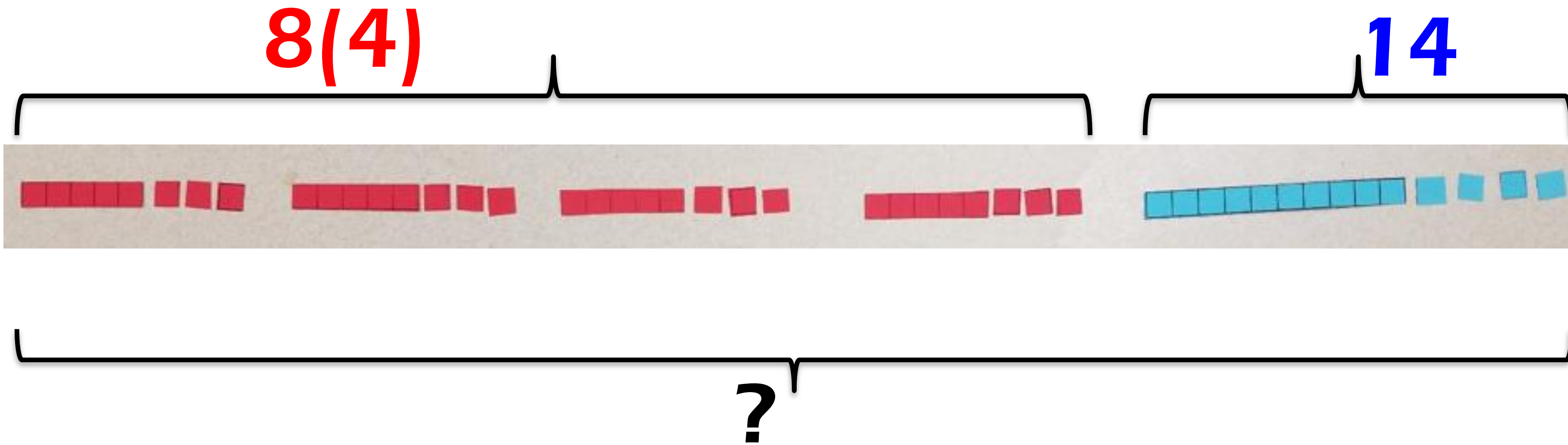
Max read 8 pages of his detective book every day. After reading for 4 days, Max still had 14 pages left. How many pages are in Max's detective book?

Problem One: Concrete Stage (Manipulatives)

Max read 8 pages of his detective book every day. After reading for 4 days, Max still had 14 pages left. How many pages are in Max's detective book?



Students could cut a base ten block in half to make two fives.
When stacked, this would connect to a ten frame; however, you lose the "every shape is a different place value in base ten" though or do you?
It would lay a foundation for work in two variables much later as x and y are both linear (to the first power), but different lengths.

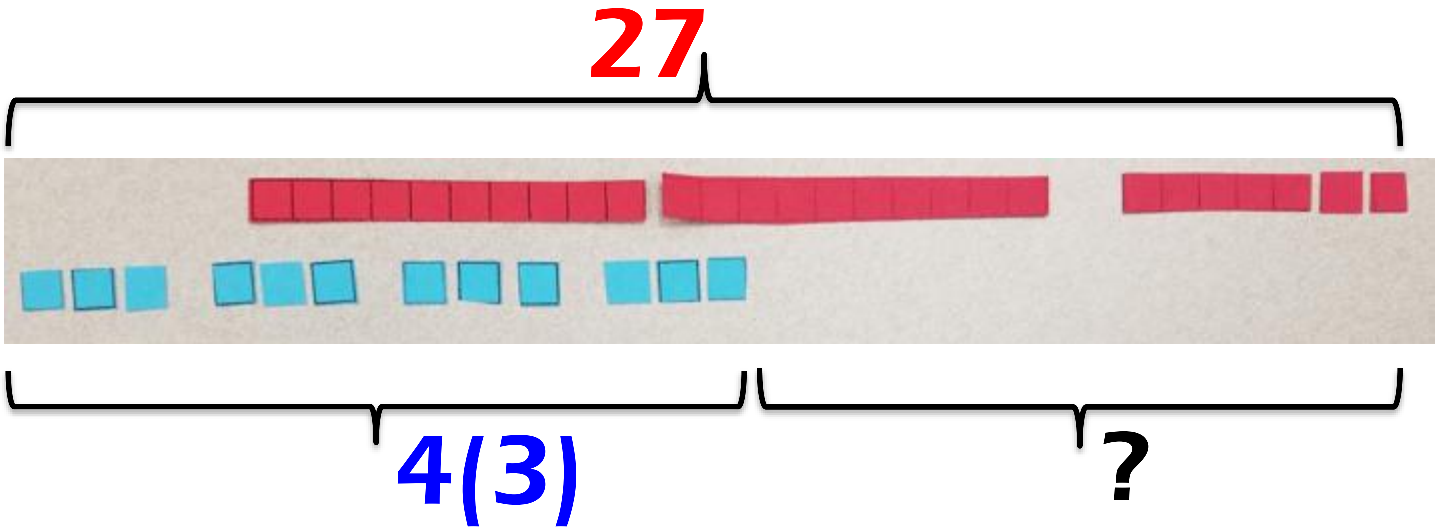
Alexa had 27 erasers.

She gave 3 erasers to each of her 4 friends.

How many erasers did Alexa have left?

Problem Two: Concrete Stage (Manipulatives)

Alexa had 27 erasers. She gave 3 erasers to each of her 4 friends. How many erasers did Alexa have left?

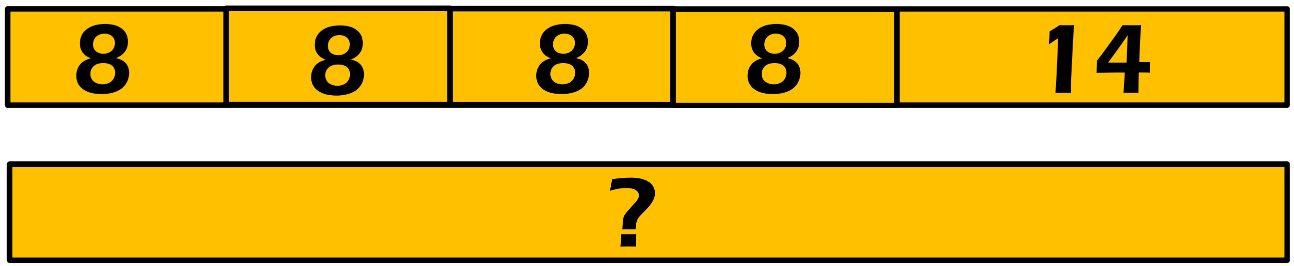


Counters could be removed from the red group; however, by representing the removed amount with the blue group you are laying a foundation for negative numbers later. This also represents the equality allowing it to be written as one equation rather than two subsequent operations with expressions. The blue group here has been stretched to see the three groups notice the group ends at the same value as the red at 12.

Representational Stage (scale & not to scale)

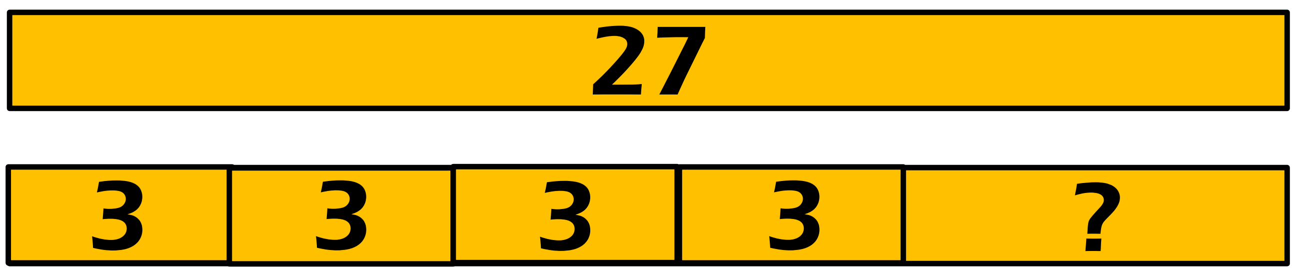
Max read 8 pages of his detective book every day. After reading for 4 days, Max still had 14 pages left. How many pages are in Max's detective book?

Step 1



Alexa had 27 erasers. She gave 3 erasers to each of her 4 friends. How many erasers did Alexa have left?

Step 1



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Connecting to the Abstract Stage

Max read 8 pages of his detective book every day. After reading for 4 days, Max still had 14 pages left. How many pages are in Max's detective book?

Step 1

8	8	8	8	14

?

$4(8) + 14 = ?$ or $8(4) + 14 = ?$

Alexa had 27 erasers. She gave 3 erasers to each of her 4 friends. How many erasers did Alexa have left?

27

Step 1

3	3	3	3	?

$27 = 4(3) + ?$ or $27 - 3(4) = ?$

1. Draw an equal sign
2. On one side, draw the values represented on one line.
3. On the other side, draw the values represented by the other line.

They are the same length!

Notice the example of the Commutative Property in play here.

4(8) versus 8(4):
4 groups of 8 or
8, 4 times

Connecting to the Abstract Stage with a Variable

Max read 8 pages of his detective book every day. After reading for 4 days, Max still had 14 pages left. How many pages are in Max's detective book?

Step 1

8	8	8	8	14
?				

$$4(8) + 14 = p \text{ or } 8(4) + 14 = p$$

Alexa had 27 erasers. She gave 3 erasers to each of her 4 friends. How many erasers did Alexa have left?

Step 1

27				
3	3	3	3	?

$$27 = 4(3) + n \text{ or } 27 - 3(4) = n$$

1. Draw an equal sign
2. On one side, draw the values represented on one line.
3. On the other side, draw the values represented by the other line.

They are the same length!

p = <u>number</u> of pages in Max's book
n = <u>number</u> of erasers left

Final Level of Abstraction: Full Symbolic

Max read 8 pages of his detective book every day. After reading for 4 days, Max still had 14 pages left. How many pages are in Max's detective book?

Step 1 $$8(4) + 14 = x$$

Alexa had 27 erasers. She gave 3 erasers to each of her 4 friends. How many erasers did Alexa have left?

Step 1 $$27 - 3(4) = x$$

Fourth Grade

Comparing Problem with CRA

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Concrete Stage (Manipulative)

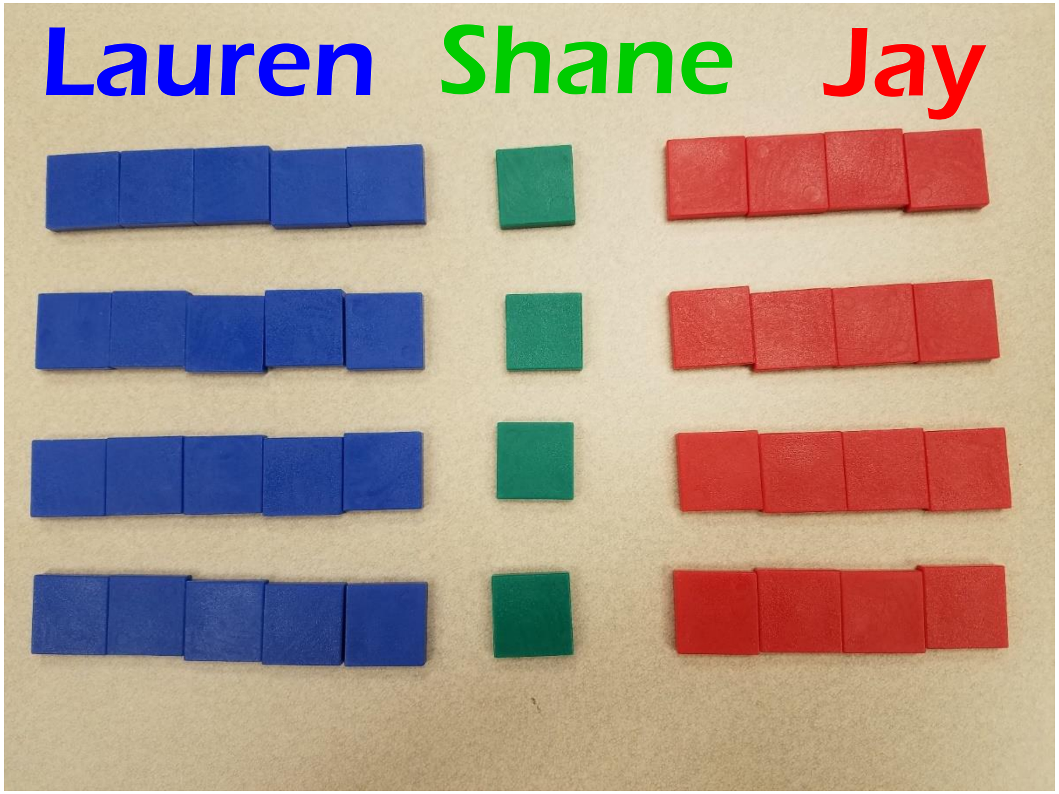
Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?

Give them each 100 crayons!

Ah, no.

Concrete Stage (Manipulative)

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



Give them each 100 crayons! Ah, no.

What if you give them each ten tiles, and had them work in groups of four?

Would they have enough to represent the entire task?

Would they have enough to engage in SMP #8 Look for and Express Regularity in Repeated Reasoning?

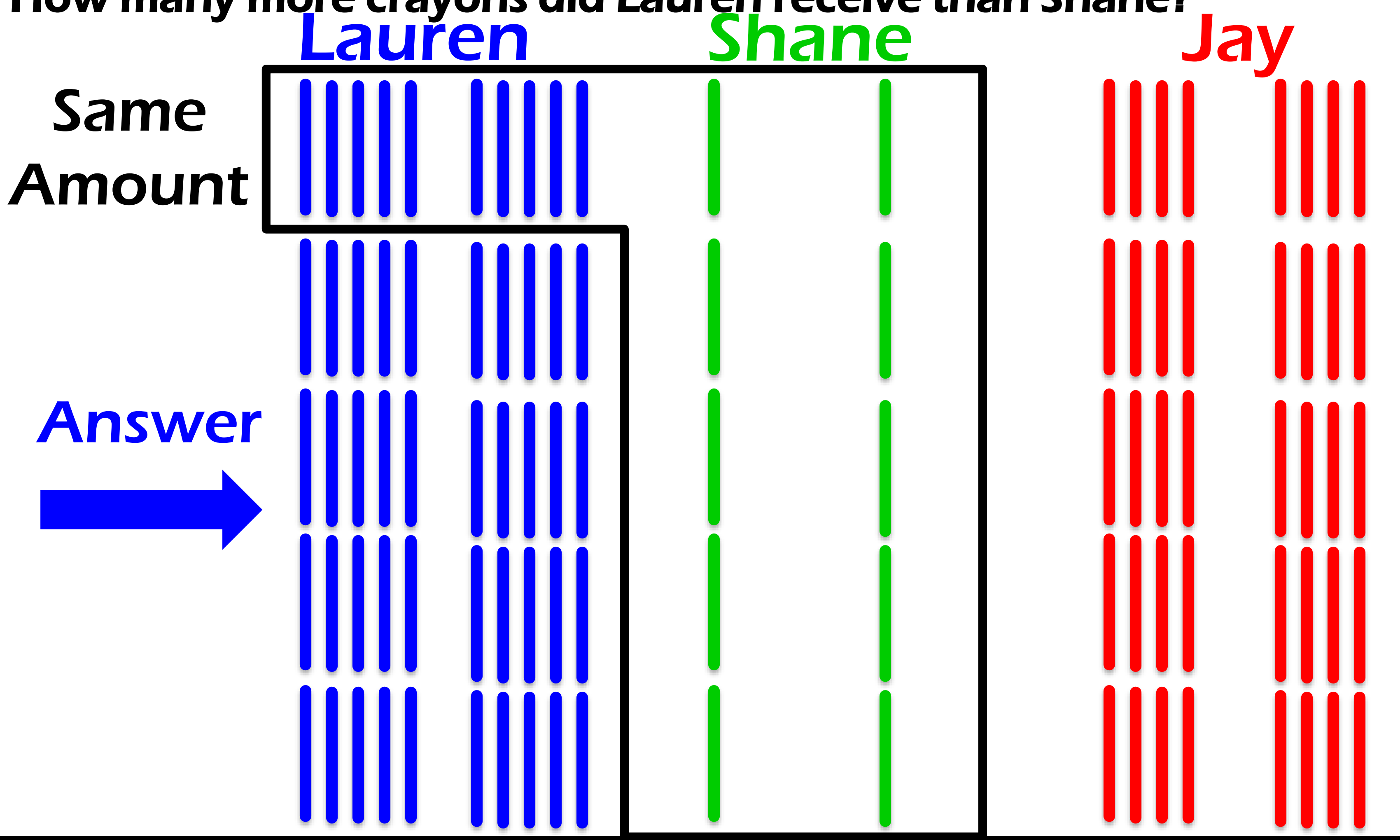
Representational Stage (Drawing)

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?

This is not 100 yet. As students, maintain these ratios (sixth grade) they use repeated groups of ten (second grade) until they make one hundred; along the way, the might utilize SMP#8, look for and express regularity in repeated reasoning, to either multiply 10 by 10 make 100 or divide 100 by 10 to make 100 (third grade).

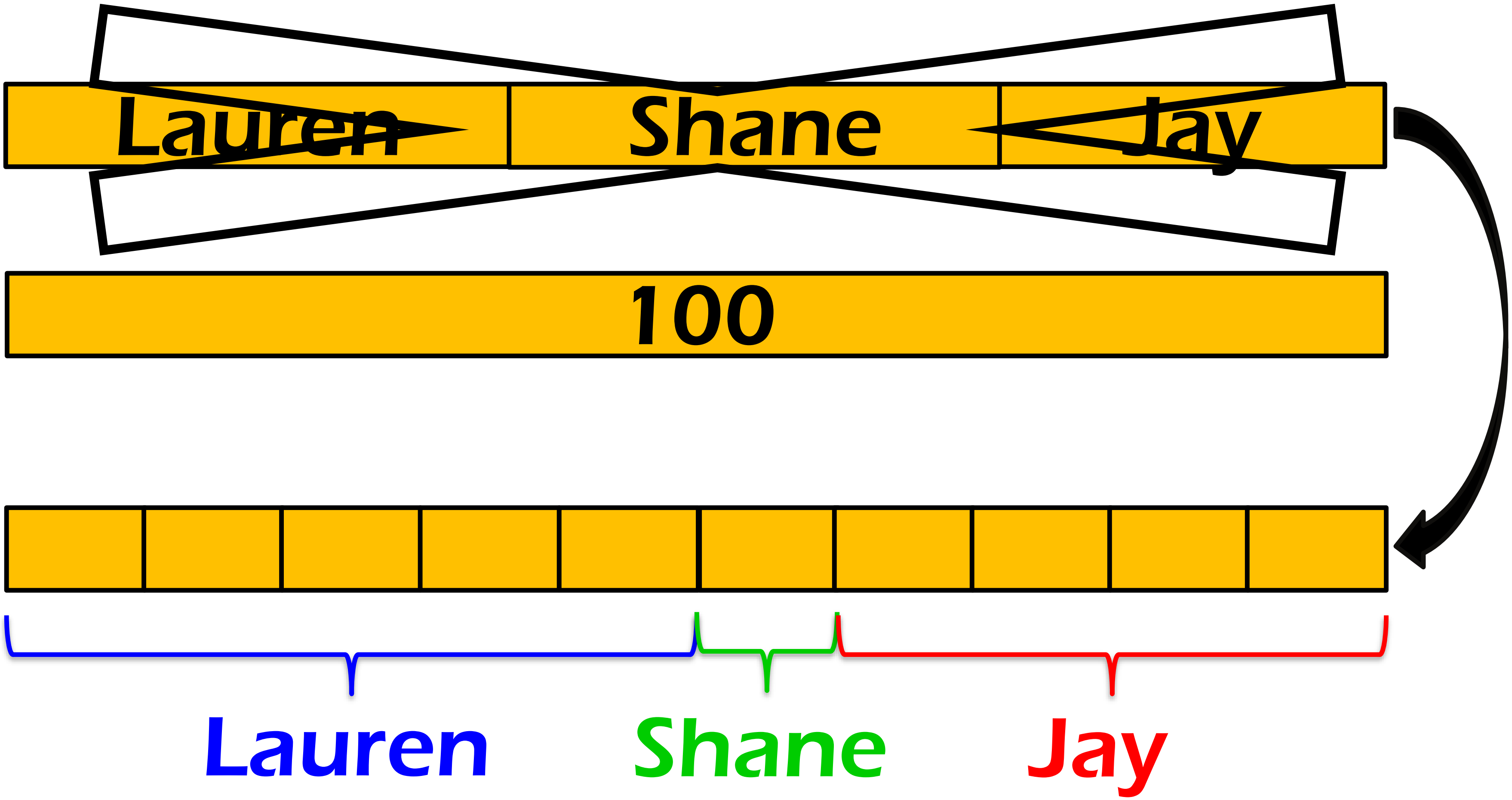
Representational Stage (Drawing)

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



Representational Stage (to scale)

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



Representational Stage (to scale)

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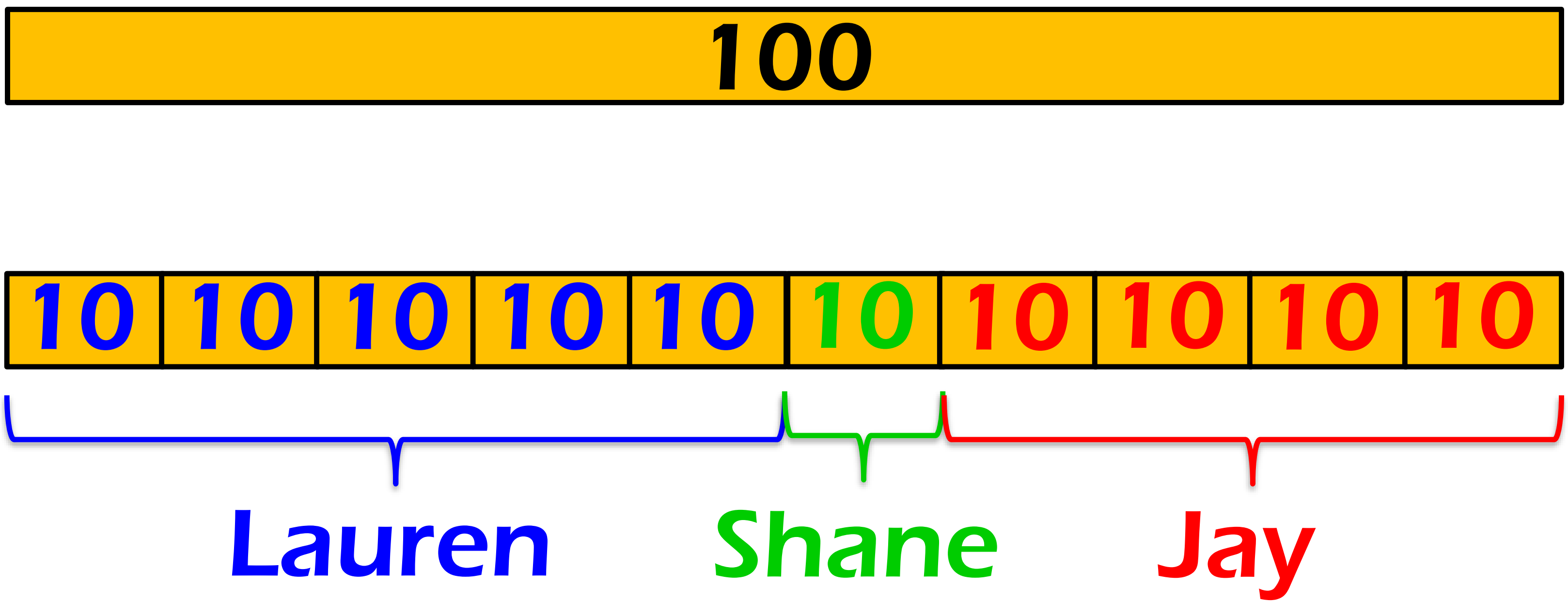

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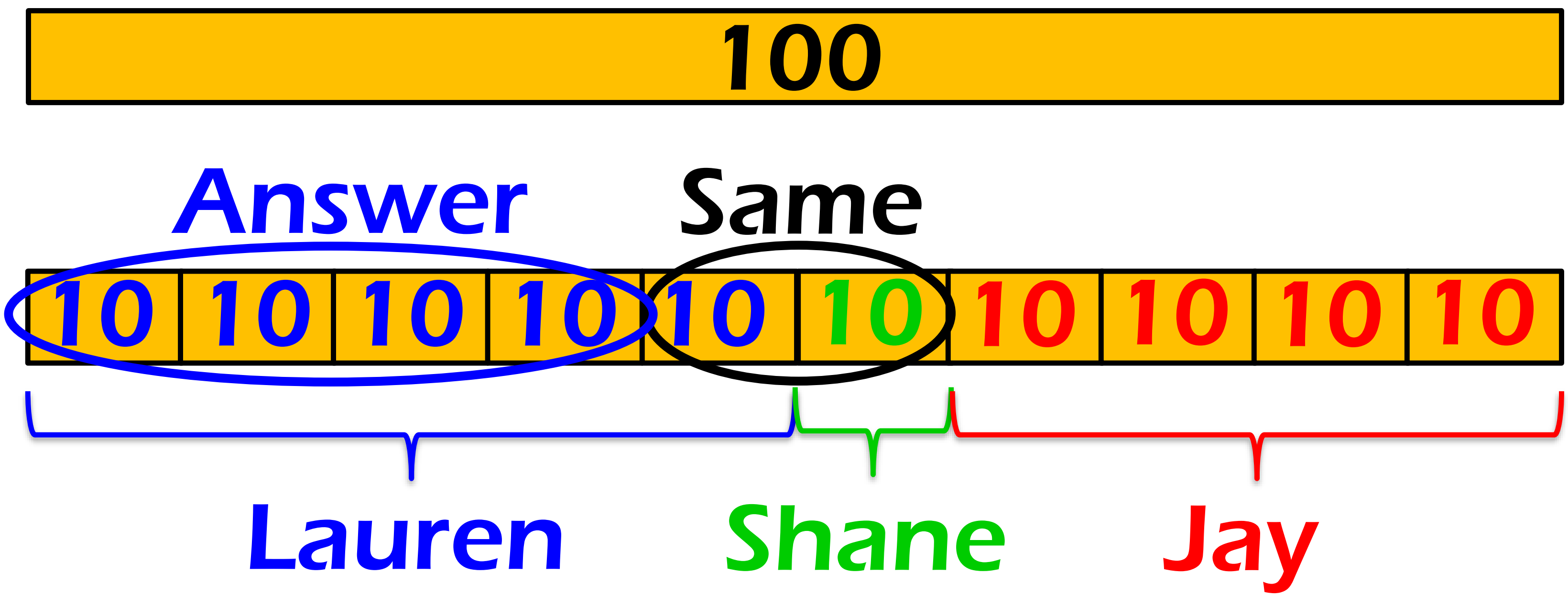
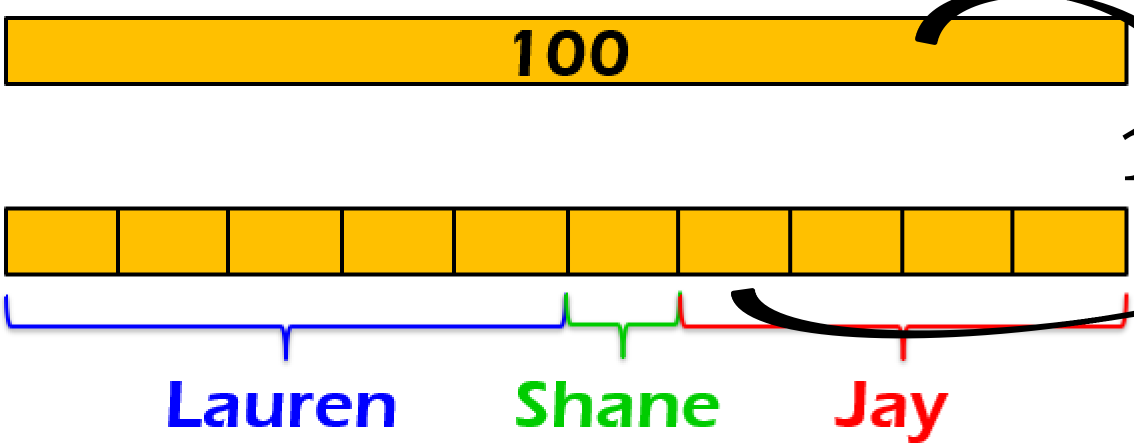


Connecting to the Abstract Stage

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



$$100 \div 10 = 10 \text{ or}$$

$$10(?) = 100$$

$$S = 10$$

$$L = 50$$

$$L - S = Answer$$

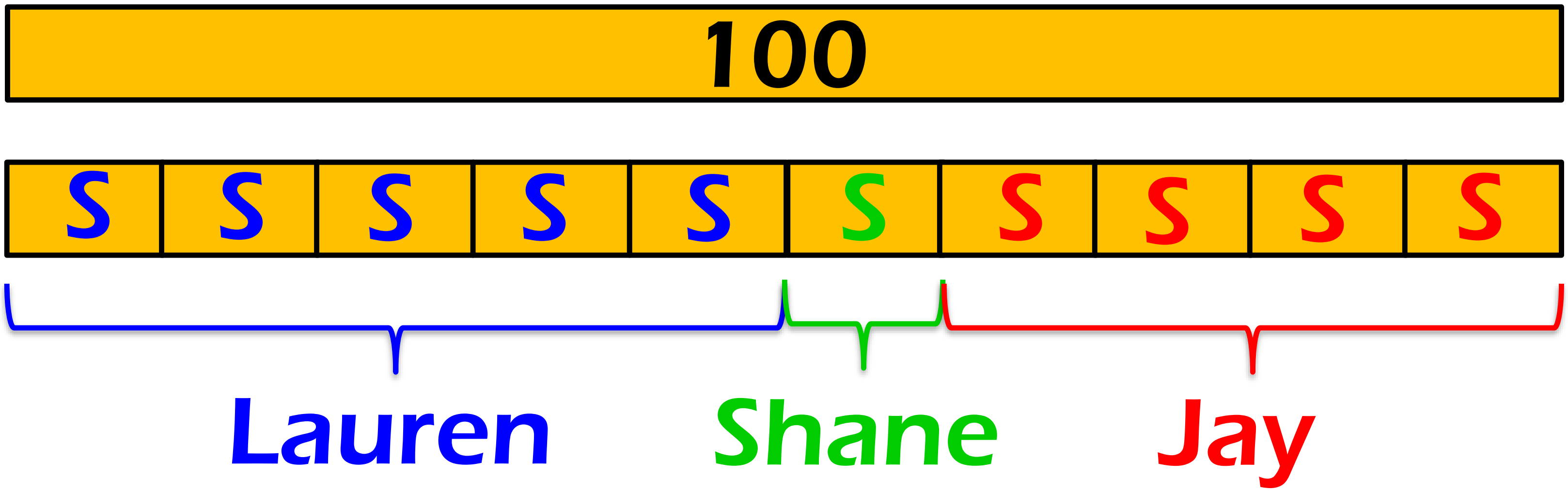
$$50 - 10 = 40$$

1. Draw an equal sign
2. On one side, draw the values represented on one line.
3. On the other side, draw the values represented by the other line.

They are the same length!

Representational Stage with a Algebra

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



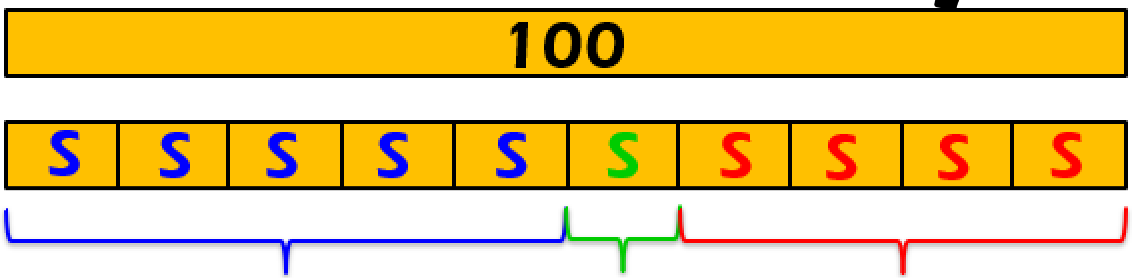
$L = 5S$ $\quad S \quad$ $J = 4S$

Notice how the context and the tape diagram both support understanding the three equations here.

Connecting to the Abstract Stage

BONUS

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



$$100 = L + S + J$$
$$100 = 5s + s + 4s$$

Lauren $L = 5S$ **Shane** S **Jay** $J = 4S$

1. **Draw an equal sign**
2. **On one side, draw the values represented on one line.**
3. **On the other side, draw the values represented by the other line.**

They are the same length!

Notice the 100 is written on the left. You want to avoid always writing it the same way so students develop procedural fluency (apply it flexibly).

$$100 = 5s + s + 4s$$

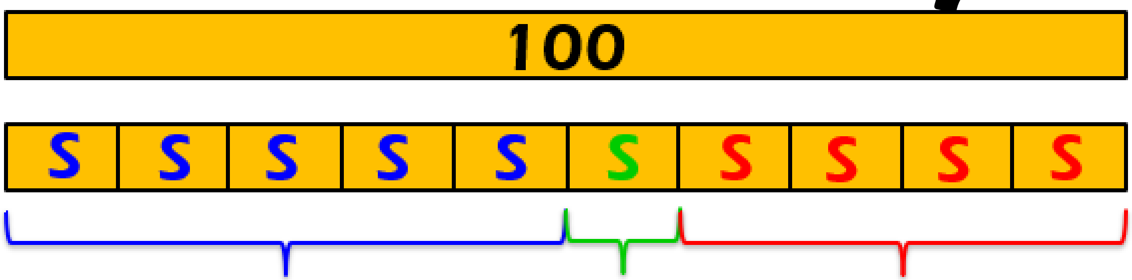
is equivalent to:

$$5s + 1s + 4s = 100$$

Connecting to the Abstract Stage

BONUS

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?



$$100 = L + S + J$$
$$100 = 5s + s + 4s$$

Lauren $L = 5S$ **Shane** S **Jay** $J = 4S$

1. **Draw an equal sign**
2. **On one side, draw the values represented on one line.**
3. **On the other side, draw the values represented by the other line.**

They are the same length!

Notice you can clearly see that these are <u>like terms</u>, and that there are ten of them. This is also a good time to discuss that <u>s=1s</u>:

$$100 = 5s + s + 4s$$

is equivalent to:

$$100 = 5s + 1s + 4s$$

Final Level of Abstraction: Full Symbolic

BONUS

Lauren, Shane & Jay shared 100 crayons. Lauren received 5 times as many crayons as Shane. Jay received 4 times as many crayons as Shane. How many more crayons did Lauren receive than Shane?

$$\mathbf{L} + \mathbf{S} + \mathbf{J} = 100$$

Lauren	Shane	Jay
$L = 5S$	S	$J = 4S$

$$100 = 5\mathbf{s} + \mathbf{s} + 4\mathbf{s}$$

Fifth Grade

Fraction Problem with CRA

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Mattoon's MEANING 4 MEMORY

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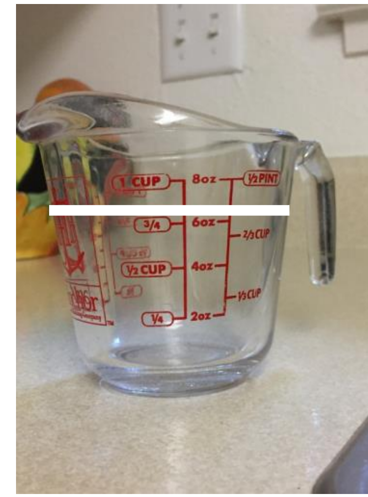
Concrete Stage (Manipulative)

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?



You could actually measure each and pour them into a single measuring cup. Yes, this would take time, but think of the conceptual understanding you would build, the questions you could ask, and the formative assessment you would gain!
Questions like, "How much of milk is in the cup after you combine them?" You have also created THE HEADACHE!

MTBoS.org

Supporting the math-twitter blog-o-sphere

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If Math Is The Aspirin, Then How Do You Create The Headache?

By Dan Meyer • June 17, 2015 • 78 Comments

https://blog.mrmeyer.com/2015/if-math-is-the-aspirin-then-how-do-you-create-the-headache/

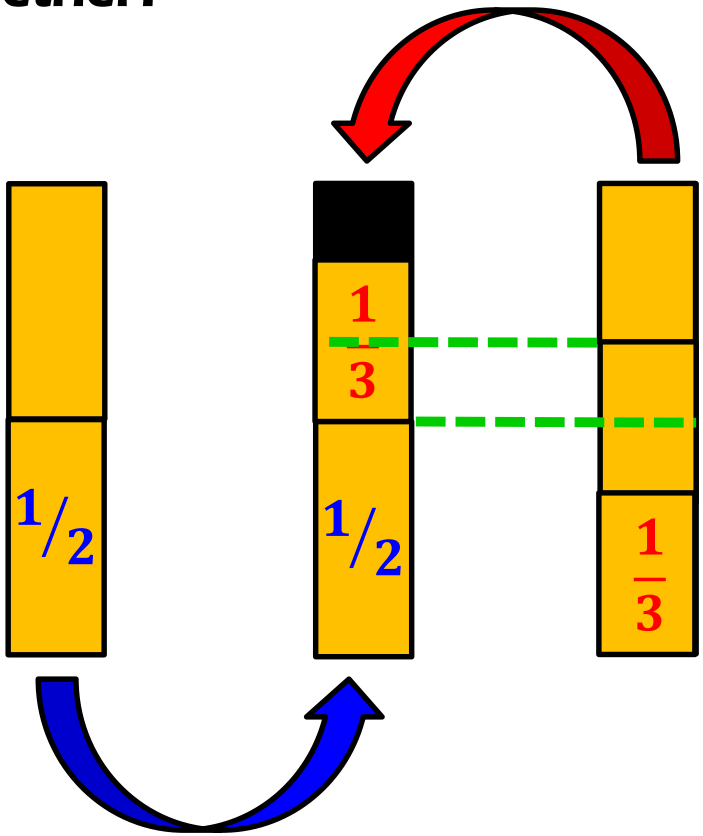
Representational Stage (Drawing)

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?

Notice the bars have been rotated (8th grade). They remain congruent and help build procedural fluency while modeling the measuring cups.

Representational Stage (Drawing)

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?



What do you notice?

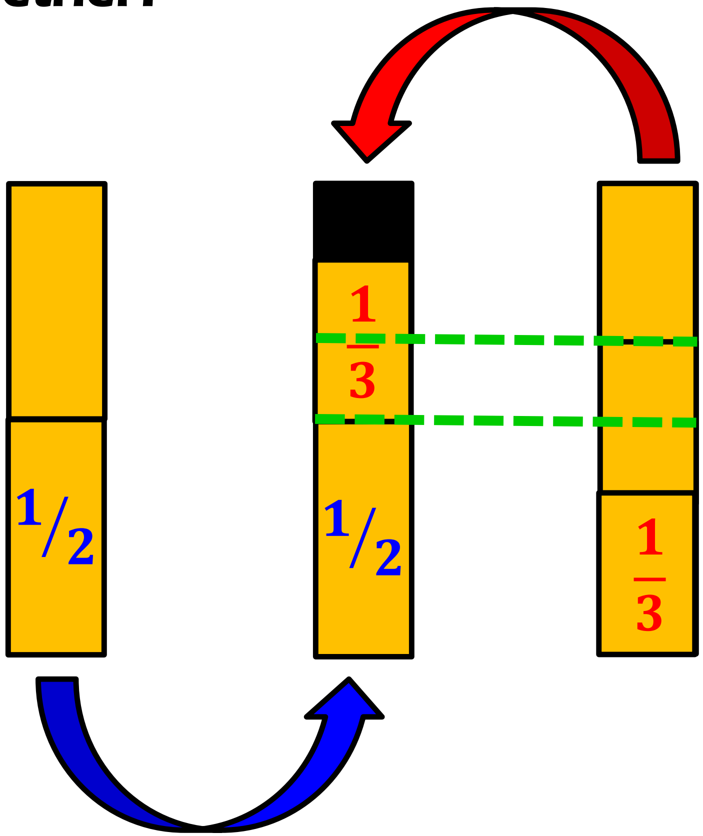
What do you wonder?

Annie Fetter

https://www.youtube.com/watch?v=a-Fth6sOaRA

Representational Stage (Drawing)

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?

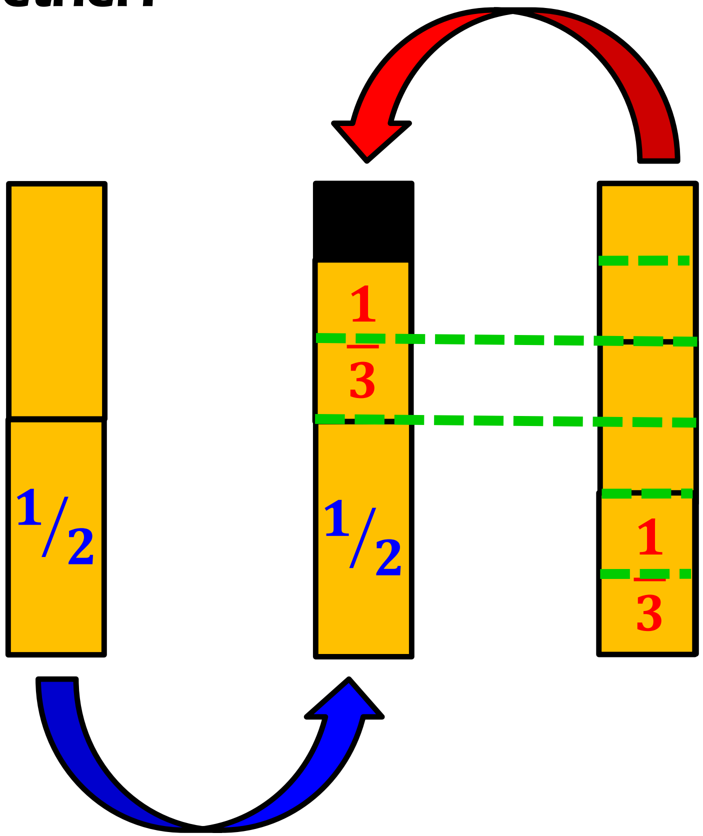


The 2/3's splits the middle box.
The 1/2 splits the right box.

How many boxes would the RIGHT bar have if each third had two boxes?

Representational Stage (Drawing)

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?



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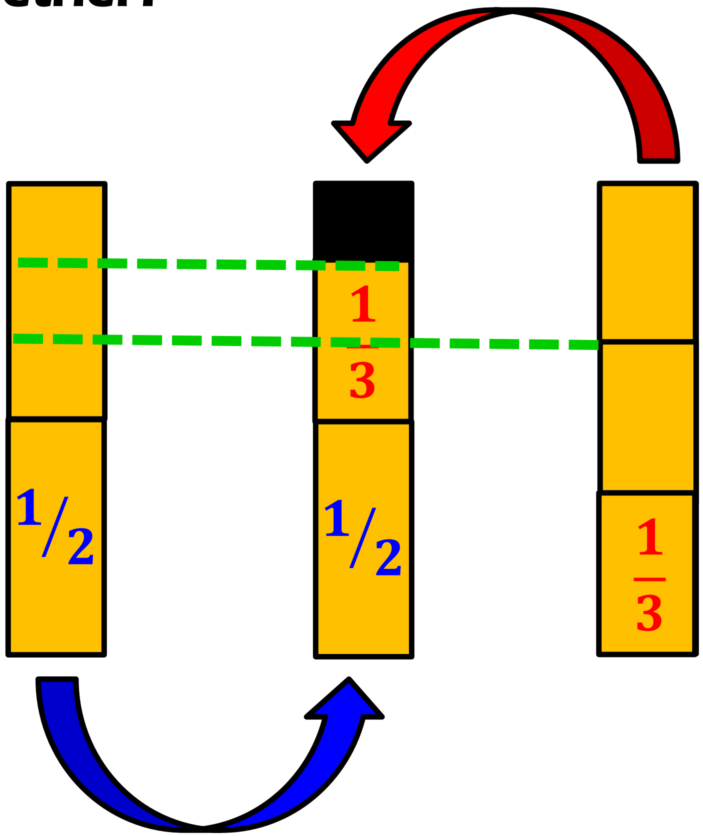

What do you notice?

What do you wonder?

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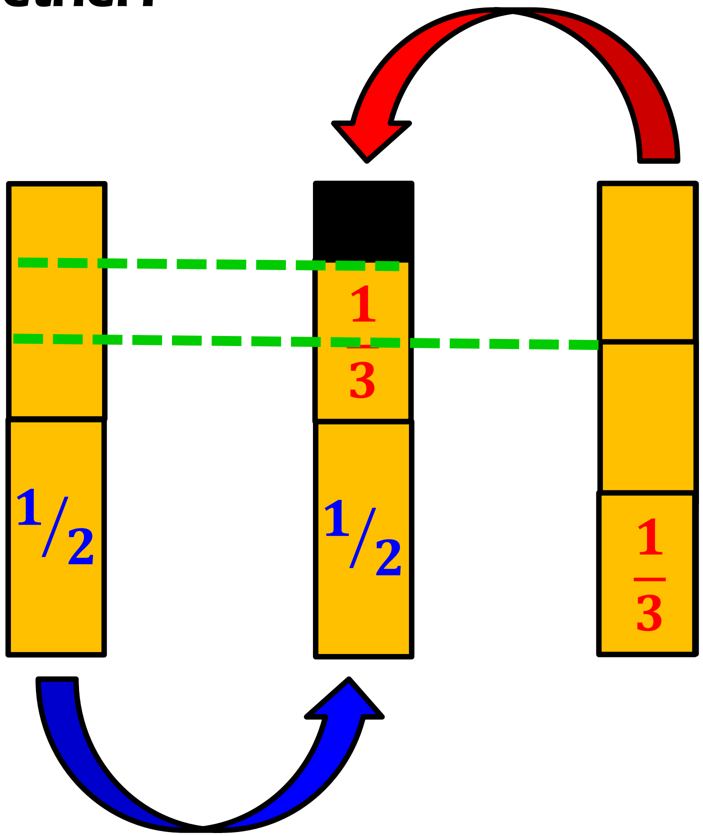


How many boxes would the LEFT bar have if each half had three boxes?

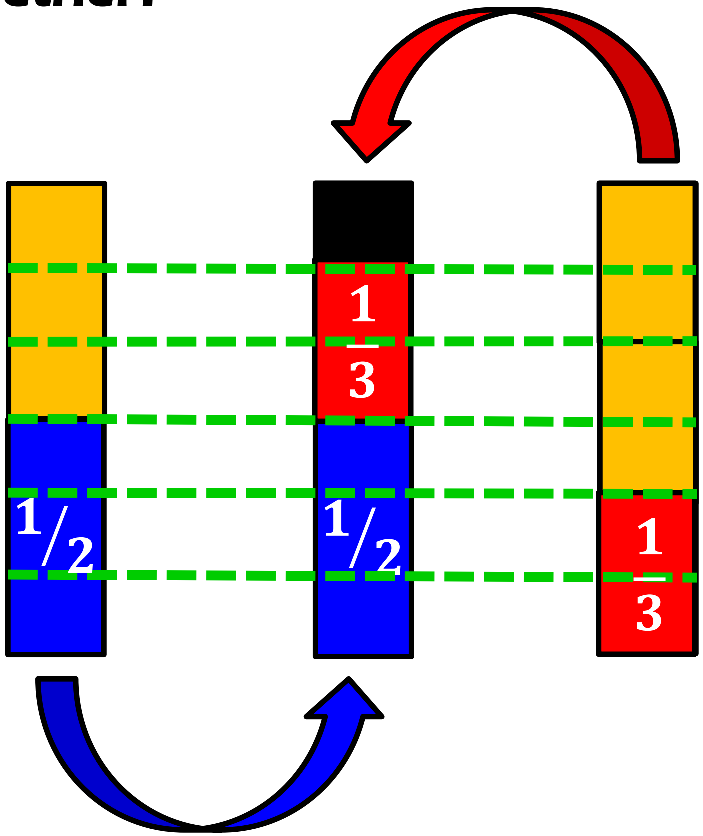
Is this the <u>same</u> or <u>different</u> than what you found for the RIGHT bar?

Same = Thumbs Up!

Different = Thumbs Down!

Representational Stage (Drawing)

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?



Each bar now has six boxes.

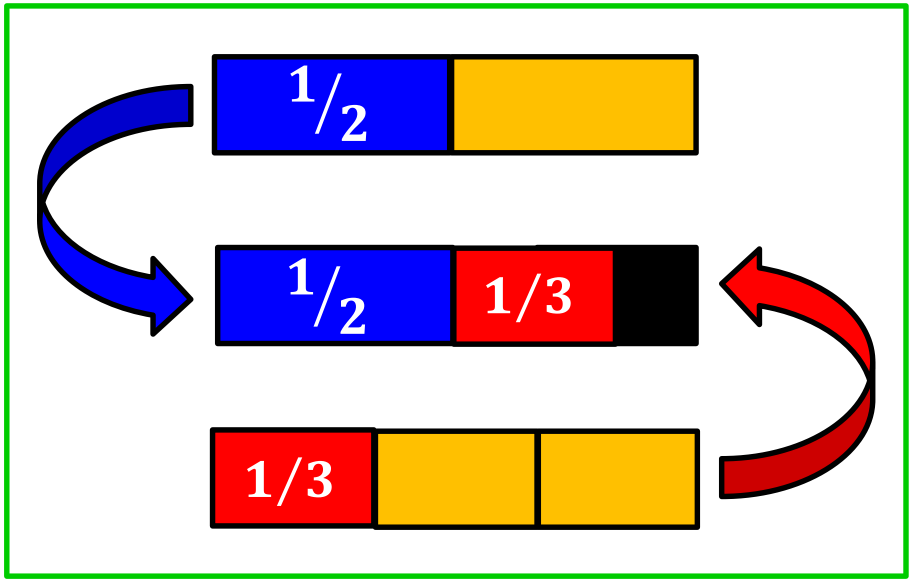
How many sixths are there in 1/2? Tell your partner.

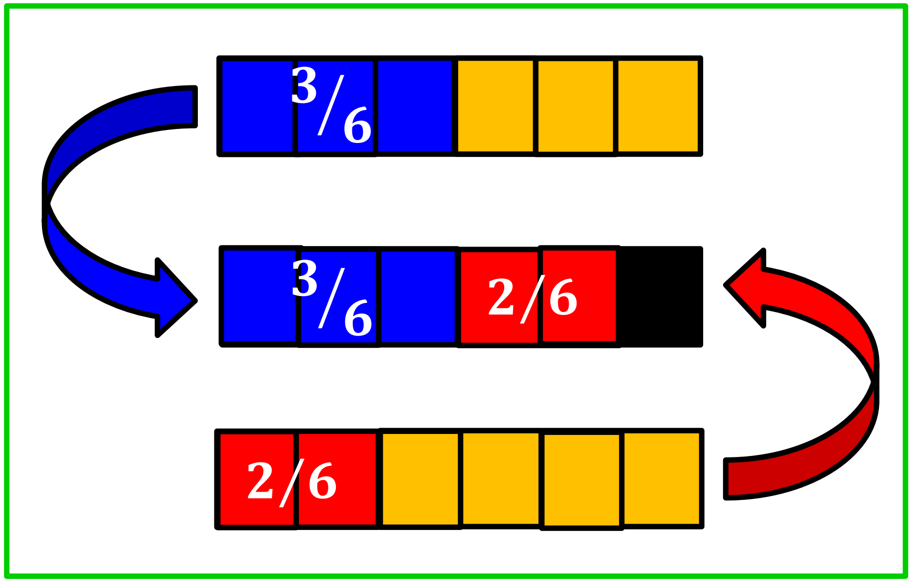
How many sixths are there in 1/3? Tell your partner.

How many total sixths in the middle bar?

Representational Stage (to scale)

Bob the Baker needs 1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?





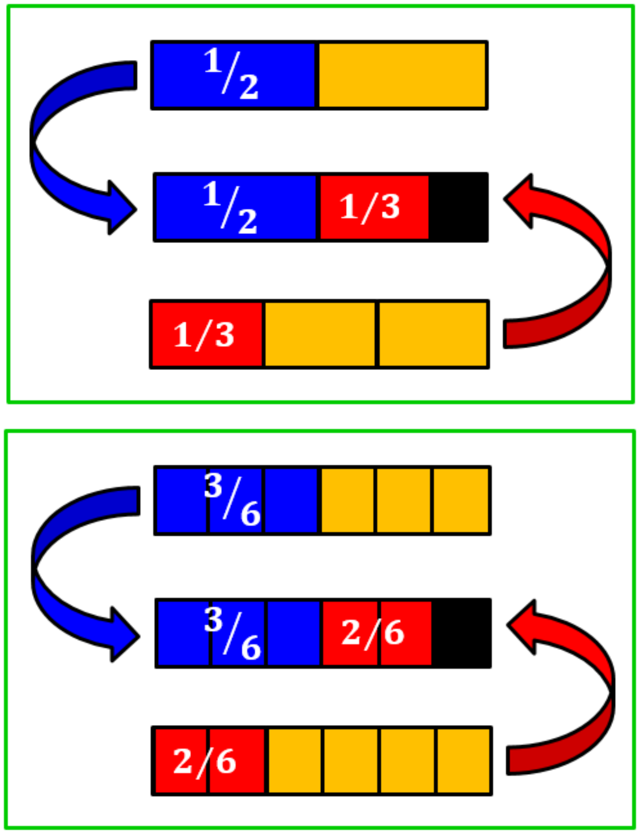
How are the two sides the same?

How are they different?

What does that tell you? It tells me that ______.

Connecting to the Abstract Stage

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?



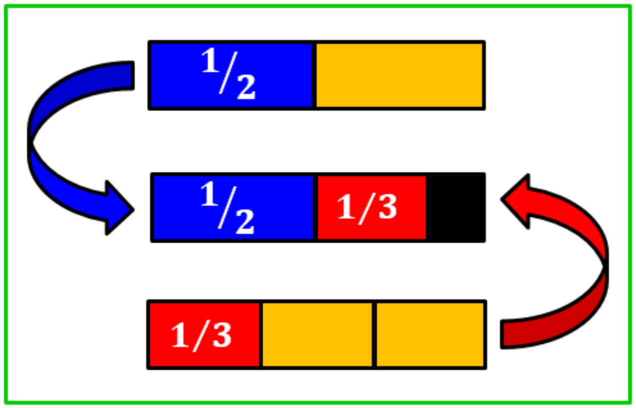
$$\frac{1}{2} + \frac{1}{3} = ?$$

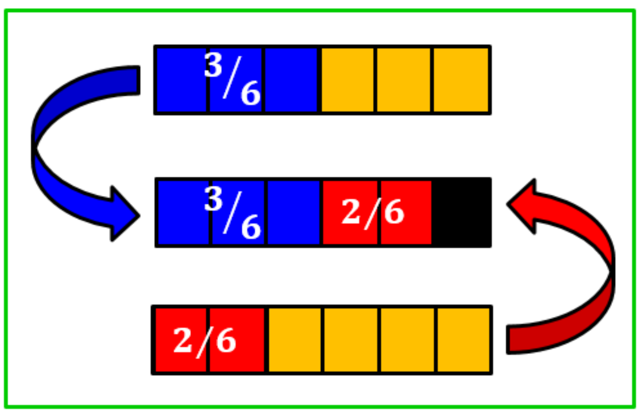
$$\frac{(3)1}{(3)2} + \frac{1(2)}{3(2)} = ?$$

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

Which Property Allows us to Do This?

Bob the Baker needs 1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?





$$\frac{1}{2} + \frac{1}{3} = ?$$

$$\frac{(3)\,1}{(3)\,2} + \frac{1\,(2)}{3\,(2)} = ?$$

$$\textit{Property in use?}$$

$$\frac{(3)\,1}{(3)\,2} = \frac{1}{2} \text{ or } 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Final Level of Abstraction: Full Symbolic

Bob the Baker needs1/2 cup of milk to make triple fudge cookies and 1/3 cup of milk to make sugar cookies. How much milk does Bob the Baker need altogether?

$$\frac{1}{2} + \frac{1}{3} = ?$$

Use the Identity Property of Multiplication

$$\frac{(3)1}{(3)2} + \frac{1(2)}{3(2)} = ?$$

Use the Big One to Find a Common Denominator

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

Sixth Grade

Dividing Fractions Problem

David Mattoon
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Hemet Unified

Mattoon's MEANING 4 MEMORY

@meaning4memory
meaning4memory.com

Concrete Stage (Manipulative)

How many 3/4 cup servings are in 2/3 of a cup of yogurt?

Do students even realize what is going on?

Do they know 3 fourths is more than 2 thirds ?

Got a Headache Yet?

Math is the Aspirin!

How Do You Know?

How many 3/4 cup servings are in 2/3 of a cup of yogurt?

Do students even realize what is going on?
Do they know 3 fourths is more than 2 thirds ?
Why is this true?
What evidence can you provide?

Concrete Stage (Manipulative)

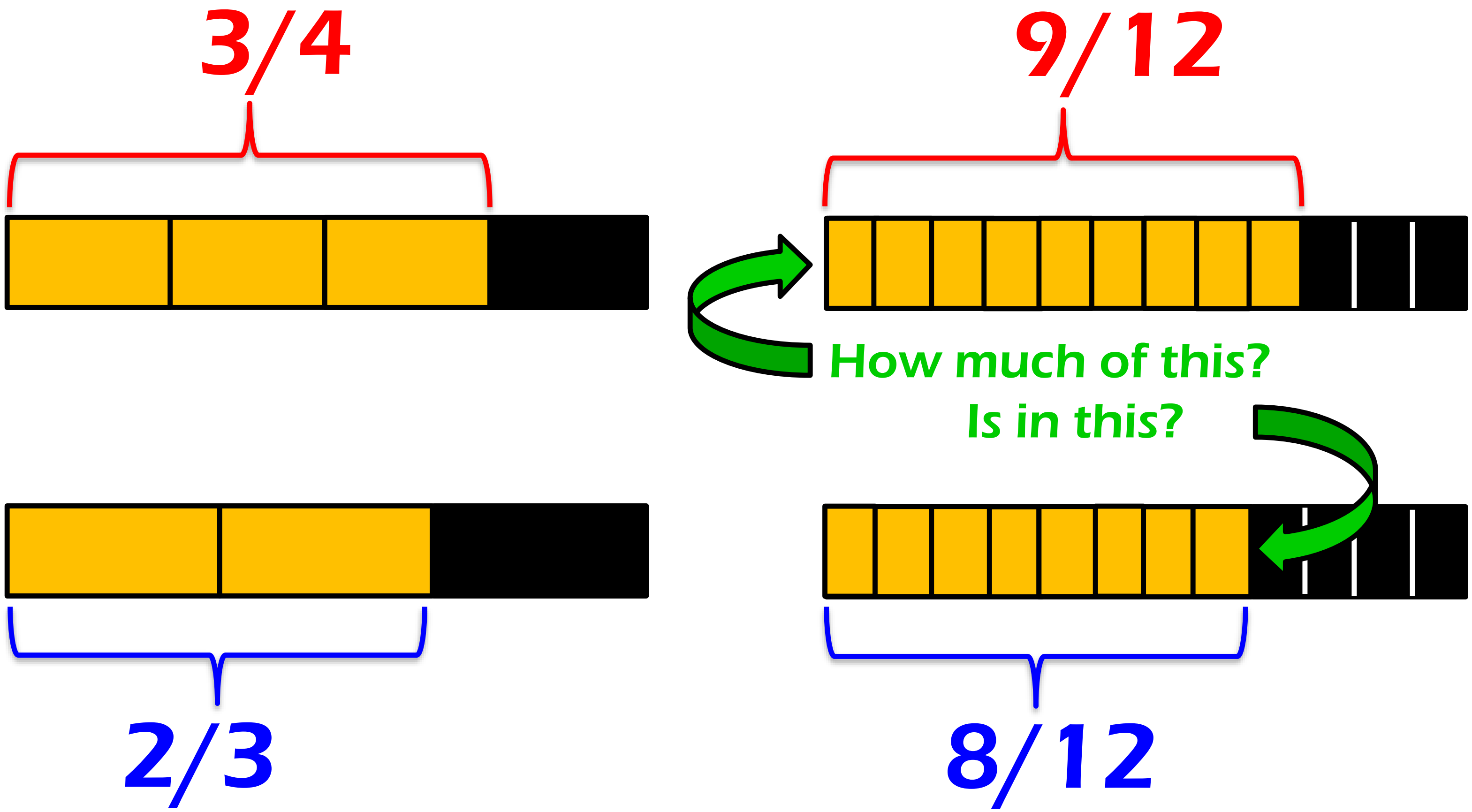
How many 3/4 cup servings are in 2/3 of a cup of yogurt?

Do they know 3 fourths is more than 2 thirds ? Why is this true? What evidence can you provide?

Two ways might be to draw a double number with two equal parts or think of it as a unit fraction away from 1.

Representational Stage (Drawing)

How many 3/4 cup servings are in 2/3 of a cup of yogurt?



Cut the fourths into thirds.

Cut the thirds into fourths.

To find the common denominator of twelfths.

Does This Always Work?

You can multiply across.

Can you divide across?

$$\frac{4}{9} \div \frac{2}{3} = ?$$

$$\frac{4 \div 2}{9 \div 3} \stackrel{?}{=} \frac{2}{3}$$

Does This Always Work?

Why don't we teach students to divide across. They love to multiply across. They love it so much they add across incorrectly…

$$\frac{4}{9} \div \frac{2}{3} = ?$$

$$\frac{4 \div 2}{9 \div 3} = \frac{2}{3}$$

Does This Always Work?

You can multiply across.

Can you divide across?

$$\frac{2}{3} \div \frac{3}{4} = ?$$

$$\frac{2 \div 3}{3 \div 4} = ?$$

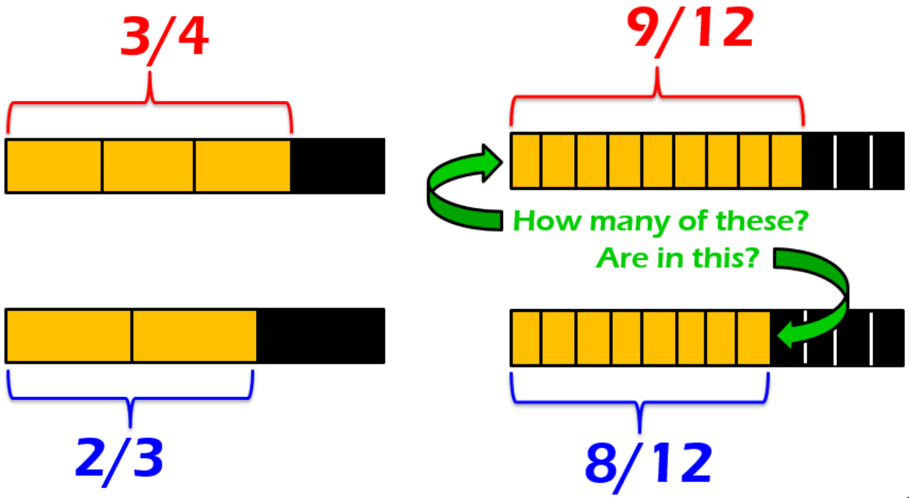
Oh; that's why we don't teach this method…

But wait!

Let's go back to our example and the pictures.

Connecting to the Abstract Stage

How many 3/4 cup servings are in 2/3 of a cup of yogurt?



$$\frac{2}{3} \div \frac{3}{4} = ?$$

Cut the thirds into fourths.

Cut the fourths into thirds.

$$\frac{(4)2}{(4)3} \div \frac{3(3)}{4(3)} = ?$$

To find the common denominator of twelfths.

$$\frac{8}{12} \div \frac{9}{12} = ?$$

$$\frac{8 \div 9}{12 \div 12} = ?$$

You can multiply across.
Can you divide across?
Yes, all the time, if you use a common denominator.

$$\frac{8/9}{12/12} = \frac{8/9}{1} = 8/9$$

Final Level of Abstraction: Full Symbolic

How many 3/4 cup servings are in 2/3 of a cup of yogurt?

$$\frac{2}{3} \div \frac{3}{4} = ?$$

Another way to divide fractions:

$$\frac{(4)2}{(4)3} \div \frac{3(3)}{4(3)} = ?$$

Find a Common Denominator

$$\frac{8}{12} \div \frac{9}{12} = ?$$

Divide Across

$$\frac{8 \div 9}{12 \div 12} = ?$$

$$\frac{8/9}{12/12} = \frac{8/9}{1} = 8/9$$

"Ours it not to Reason Why; Just Invert & Multiply"

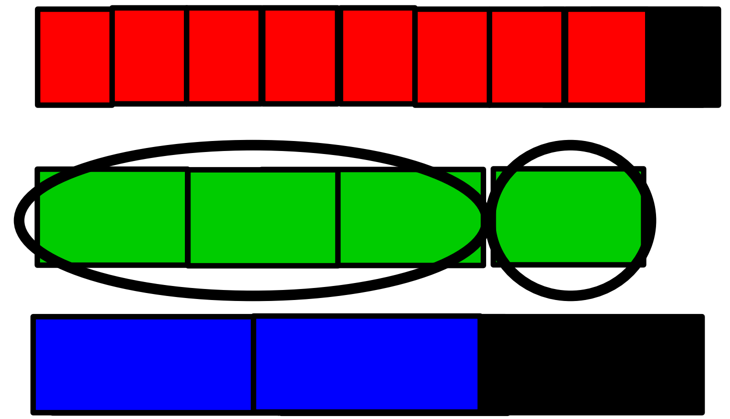
$$\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$$

or

$$\frac{2}{3} \cdot \frac{4}{3} = \frac{8}{9}$$

8/9

4/3 of 2/3



2/3

4 thirds of 2 thirds is equivalent to 8 ninths

Seventh Grade

Proportion & Percent Problems

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Mattoon's
MEANING 4 MEMORY

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meaning4memory.com

CA Framework is a Great Place for Problems!

Examples: Multi-Step Percent Problems	7.RP.3▲

1. A sweater is marked down 30%. The original price was \$37.50. What is the price of the sweater after it is marked down?

Solution: A simple diagram like the one shown can help students see the relationship between the original price, the amount taken off, and the sale price of the sweater. In this case, students can solve the problem either by finding 70% of \$37.50, or by finding 30% of \$37.50 and subtracting it.

\$37.50 Original price of sweater	
30% of 37.50	70% of 37.50 Sale price of sweater

Seeing many examples of problems such as this one helps students to see that discount problems take the form $(100\% - r\%) \cdot p = d$, where r is the amount of reduction, p is the original price, and d is the discounted price.

https://www.cde.ca.gov/ci/ma/cf/mathfwchapters.asp

What is written into the CA Framework?

Tape Diagrams!

Concrete Stage (Manipulative)

A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

Give every student a sweater and $37.50.

Just Kidding; however, pictures are great supports.

A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

Consider a double bar graph for proportionality.

Percent	
100%	
30%	70%

	Value	
Original Price	$37.50	
	Discount	New Price

Students can find either the new price or the discount with this model depending upon what the question is asking for.

Connecting to the Abstract Stage

A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

Consider a double bar graph for proportionality.

Percent

100%	
30%	70%

Value

$37.50	
Discount	New Price

In this case we want the new price, do you see the proportion?

Connecting to the Abstract Stage

A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

Consider a double bar graph for proportionality.

Percent		=	Value	
100%			$37.50	
30%	70%		Discount	New Price

You can write the proportion just like it looks on the tape diagram.

Connecting to the Abstract Stage

A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

Consider a double bar graph for proportionality.

Percent		=	Value	
100%			$37.50	
30%	70%		Discount	New Price

$$\frac{100}{70} = \frac{37.50}{p}$$

You can write it just like it looks on the tape diagram.

Abstract Stage

A sweater is marked down 30%. The original price was \$37.50. What is the price of the sweater after it is marked down?

If it is marked down 30%, then you pay 70%.

$$\frac{Original\ Price}{New\ Price} = \frac{100}{70} = \frac{37.50}{p}$$

$$100p = 37.5(70)$$

$$p = \frac{37.5(70)}{100}$$

How would you simplify the answer?

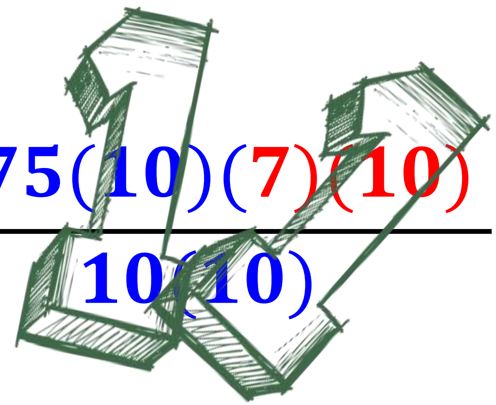
Abstract Stage

A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

If it is marked down 30%, then you pay 70%.

$$\frac{Original\ Price}{New\ Price} = \frac{100}{70} = \frac{37.50}{p}$$

$$100p = 37.5(70)$$



$$p = \frac{37.5(7\cancel{0})}{10\cancel{0}} = \frac{37.5(70)}{10(10)} = \frac{3.75(10)(7)(10)}{10(10)}$$

Why multiply it? Use procedures flexibly to build fluency.

Mark UP: New Price

A sweater is marked **UP** 30%.

The original price was $37.50.

What is the price of the sweater after it is marked **UP** ?

Consider a double bar graph for proportionality.

Percent		Value	
130%		New Price	
30%	100%	Mark Up	$37.50

$$\frac{130}{100} = \frac{p}{37.50}$$

You have to emphasize sense-making here.
The 100% is not always in the same place.

Mark UP: Amount of Mark Up

A sweater is marked UP 30%.

The original price was $37.50.

How much was the sweater **marked UP** in dollars?

Consider a double bar graph for proportionality.

Percent

130%	
30%	100%

=

Value

New Price	
Mark Up	$37.50

$$\frac{30}{100} = \frac{m}{37.50}$$

Use any proportion you need; not just top & bottom.

Eighth Grade

Linear Systems with CRA

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Mattoon's MEANING 4 MEMORY

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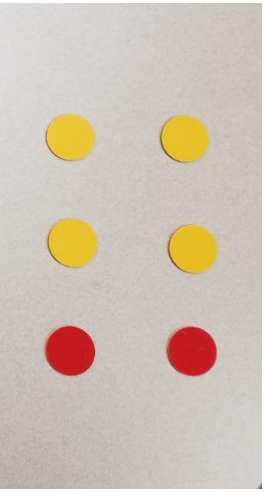
Concrete Stage (Manipulative)

Each year, Quinn plants 24 flowers in his garden. This year, he planted only red flowers and yellow flowers. Quinn prefers yellow, so he planted twice as many yellow flowers as red flowers.

Twice as many yellow as red.

Concrete Stage (Manipulative)

Each year, Quinn plants 24 flowers in his garden. This year, he planted only red flowers and yellow flowers. Quinn prefers yellow, so he planted twice as many yellow flowers as red flowers.

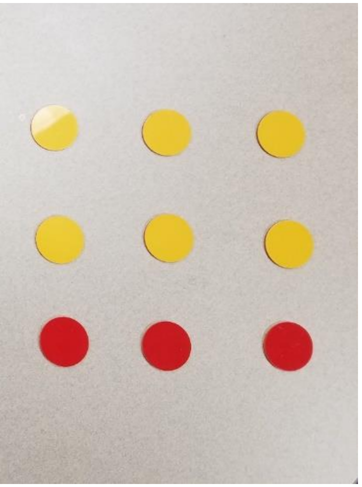


Twice as many yellow as red.

How long do you think it will take for them to realize?

Concrete Stage (Manipulative)

Each year, Quinn plants 24 flowers in his garden. This year, he planted only red flowers and yellow flowers. Quinn prefers yellow, so he planted twice as many yellow flowers as red flowers.



Twice as many yellow as red.

Which Standard of Mathematical Practice is this?

Concrete Stage (Manipulative)

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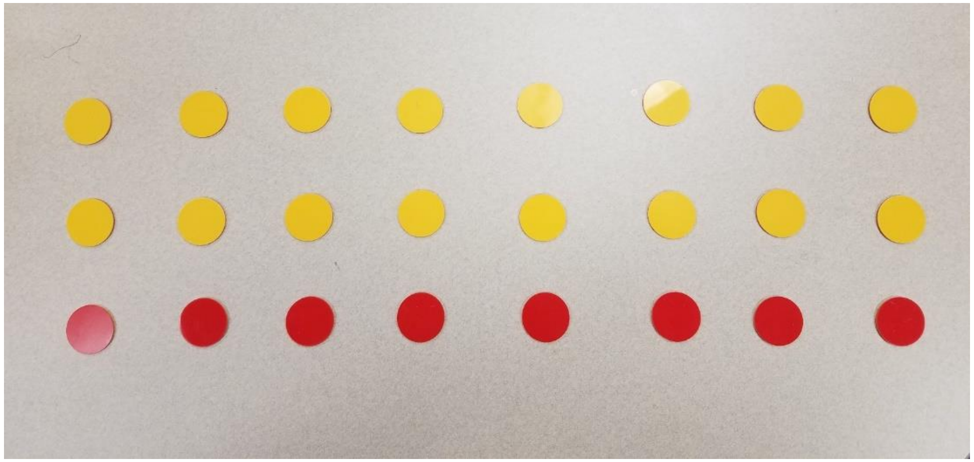


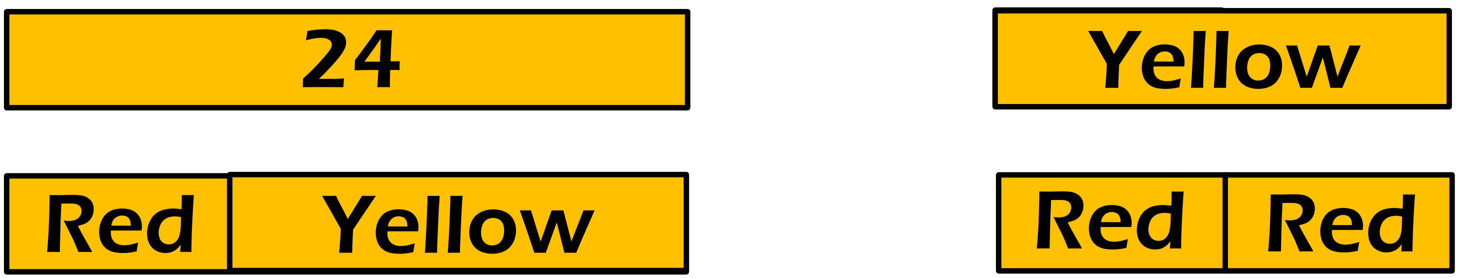
Twice as many yellow as red.

There are now 24.

Why give away the answer at this stage?

Representational Stage (to scale)

Each year, Quinn plants 24 flowers in his garden. This year, he planted only red flowers and yellow flowers. Quinn prefers yellow, so he planted twice as many yellow flowers as red flowers.



They could draw flowers (circles) to solve the problem; however, tape diagrams can help students understand solving linear systems by substitution.

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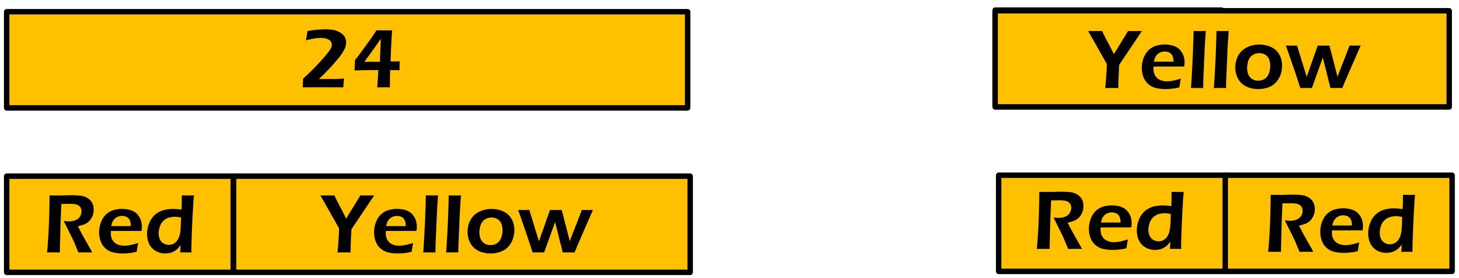

Some students can substitute into expressions, but do not fully understand what is happening so they struggle with solving systems by linear substitution.

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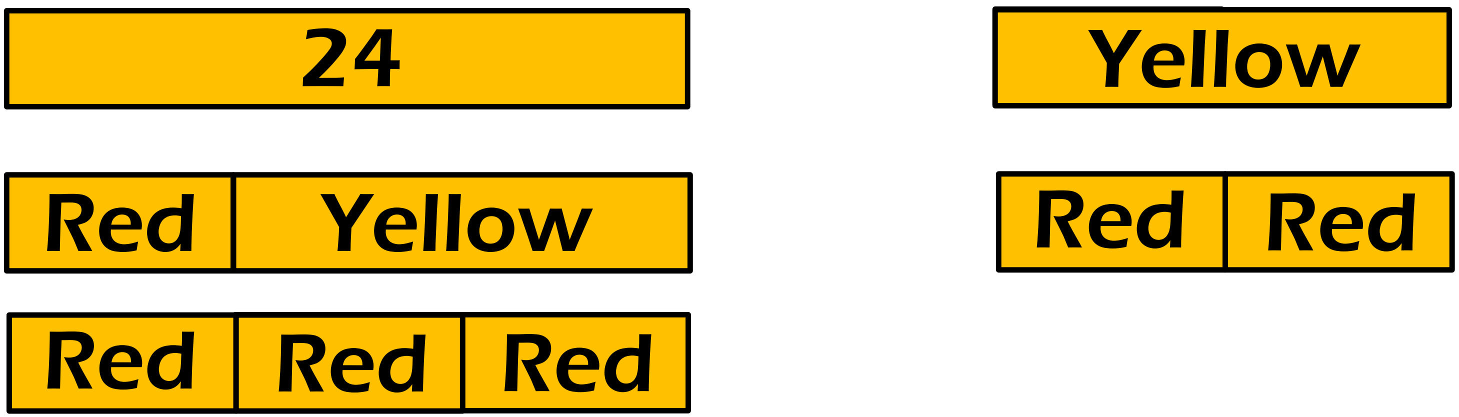

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Connecting to the Abstract Stage

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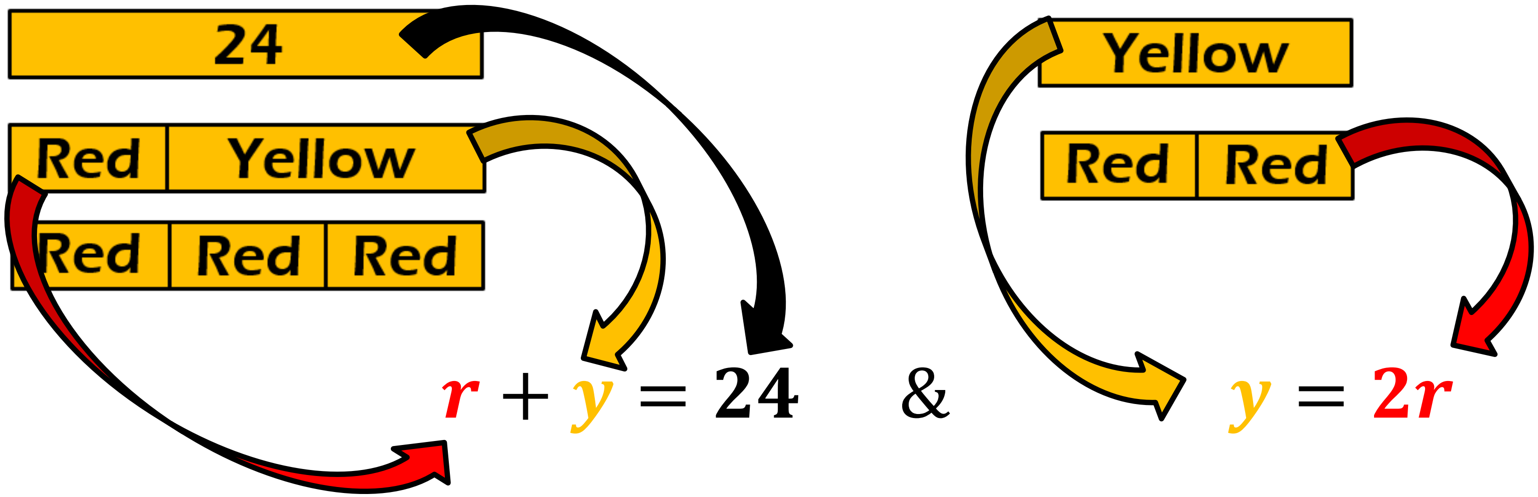

$r + y = 24$ & $y = 2r$

Write the system.

Connecting to the Abstract Stage

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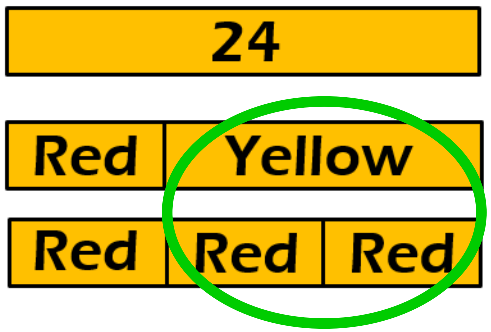

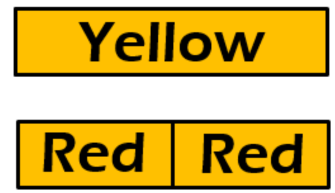

$$r + y = 24 \quad \& \quad y = 2r$$

$$r + 2r = 24$$

Realize that one yellow has the same number of flowers as two reds, and rewrite the equation with a single variable.

Abstract Stage

Each year, Quinn plants 24 flowers in his garden. This year, he planted only red flowers and yellow flowers. Quinn prefers yellow, so he planted twice as many yellow flowers as red flowers.

$$r + y = 24$$
$$y = 2r$$

$$r + 2r = 24$$
$$3r = 24$$
$$r = 8$$

$$y = 2r$$
$$y = 2(8)$$
$$y = 16$$

$$8 + 16 \overset{?}{=} 24$$
$$24 = 24$$

Solve for your one variable in the new equation and then substitute the value in to find the other variable. Don't forget to check your work.

NCTM's

Effective Teaching Practices

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Which Effective Mathematics Teaching Practices are in Play?

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

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ThinkingBlocks.com

Problems, Games & Tools

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Math Playground

Give your Brain a Workout

Examples Taken from https://www.mathplayground.com/thinkingblocks.html

Thinking Blocks

Math Playground

Give your Brain a Workout

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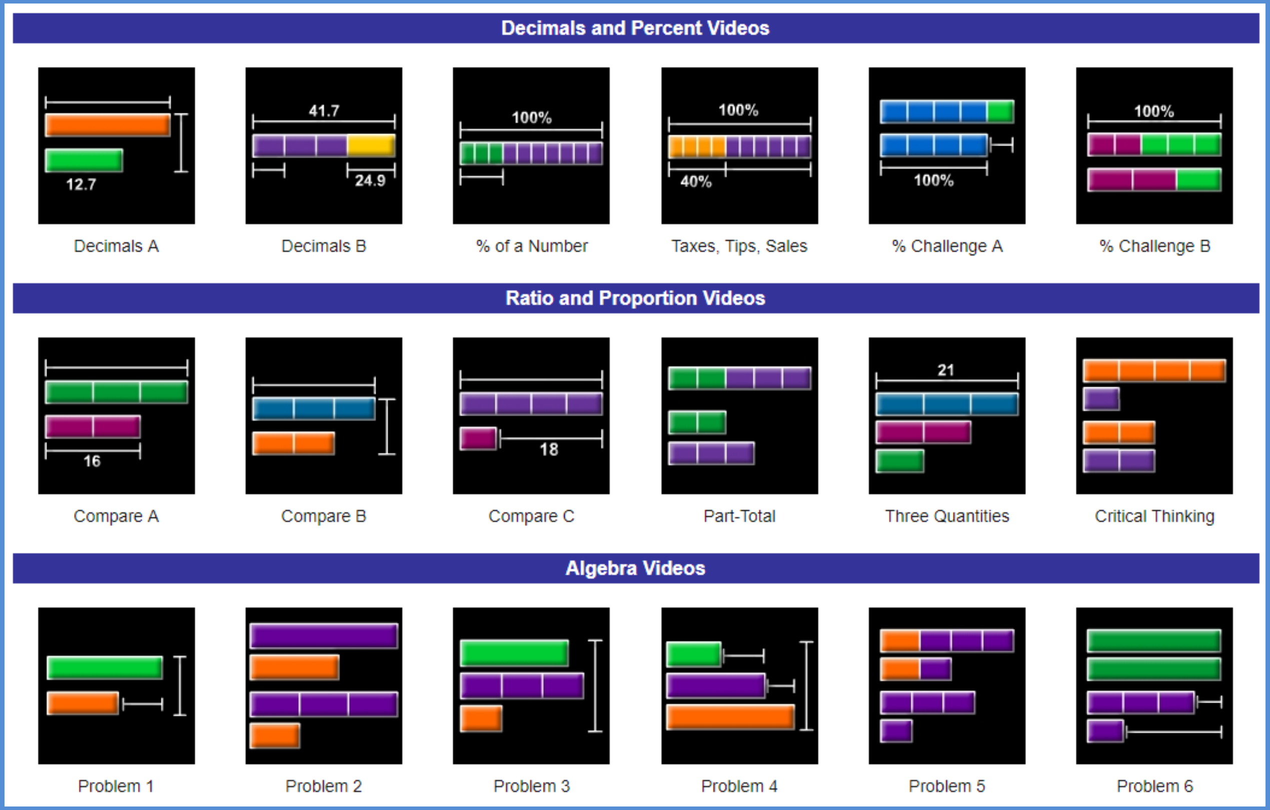
Thinking Blocks



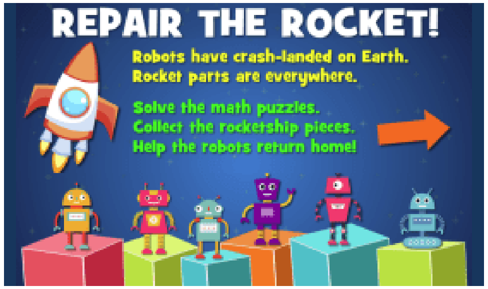
1st Grade   2nd Grade   3rd Grade   4th Grade   5th Grade   6th Grade

Introducing New and Improved Thinking Blocks®

We redesigned Thinking Blocks and packed it full of new features!

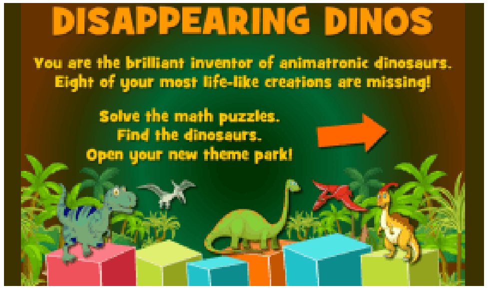
read aloud word problems - **visual prompts** - **better models** - **engaging themes** - **mobile friendly**

Thinking Blocks works well on all devices.



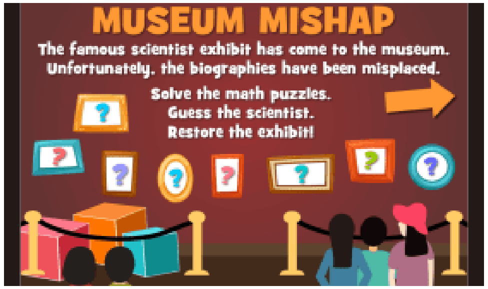
Thinking Blocks Junior

Thinking Blocks Addition



Thinking Blocks Multiplication

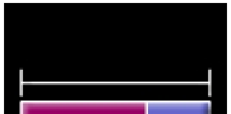
Thinking Blocks Fractions

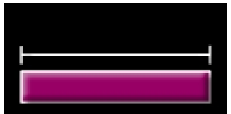


Thinking Blocks Ratios

Thinking Blocks Tool

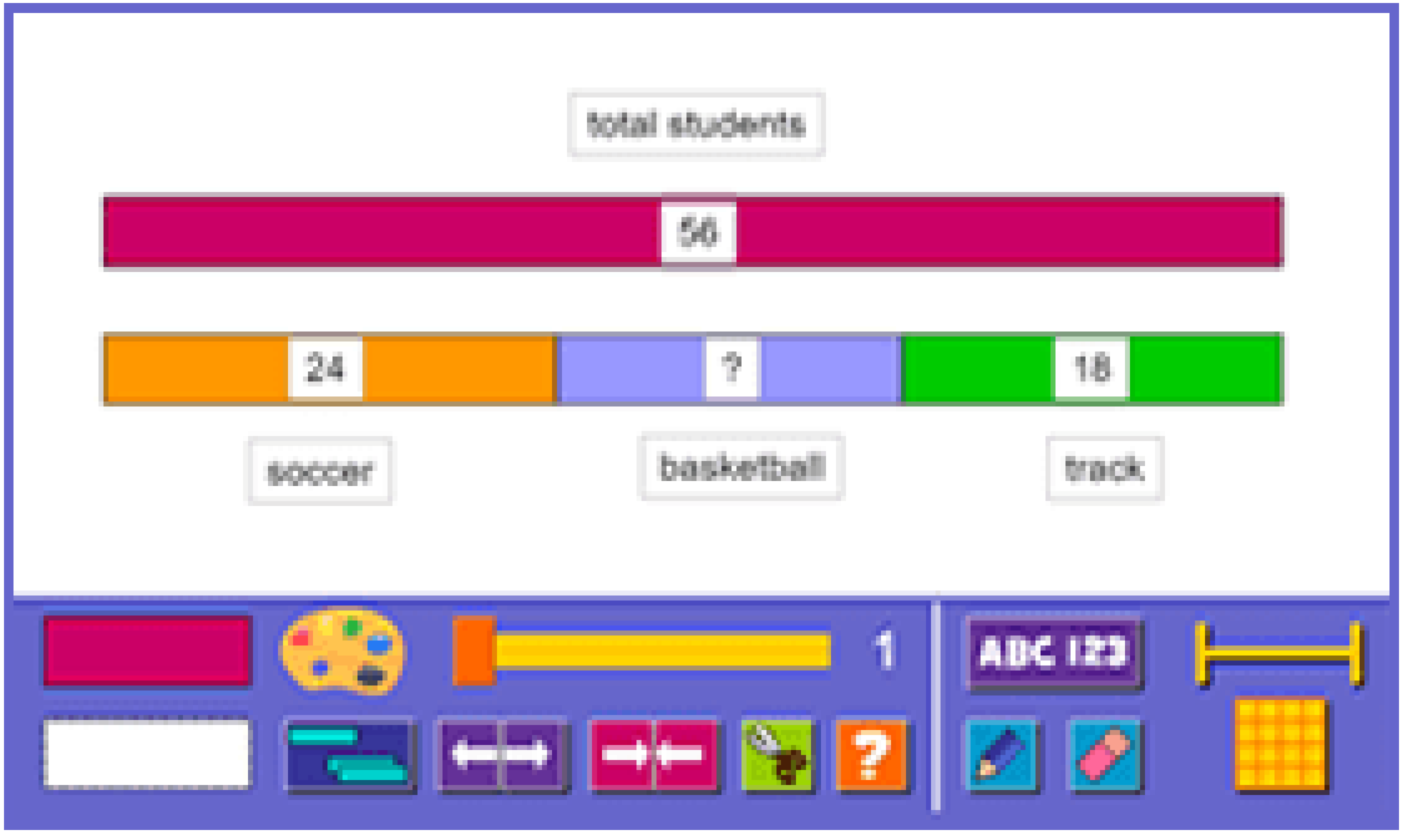
Addition and Subtraction Videos



An Online Tool is Available to Create Your Own

https://www.mathplayground.com/thinking_blocks_modeling_tool/index.html



Thinking Blocks Tool

Want More *Meaning* - *full* Tips?

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Opening Problem

An Argument for Tape Diagrams

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Nicole had 7 times as many nickels as Lexi. After Nicole gave Lexi some of her nickels, each girl had 20. How many nickels had Nicole given to Lexi?

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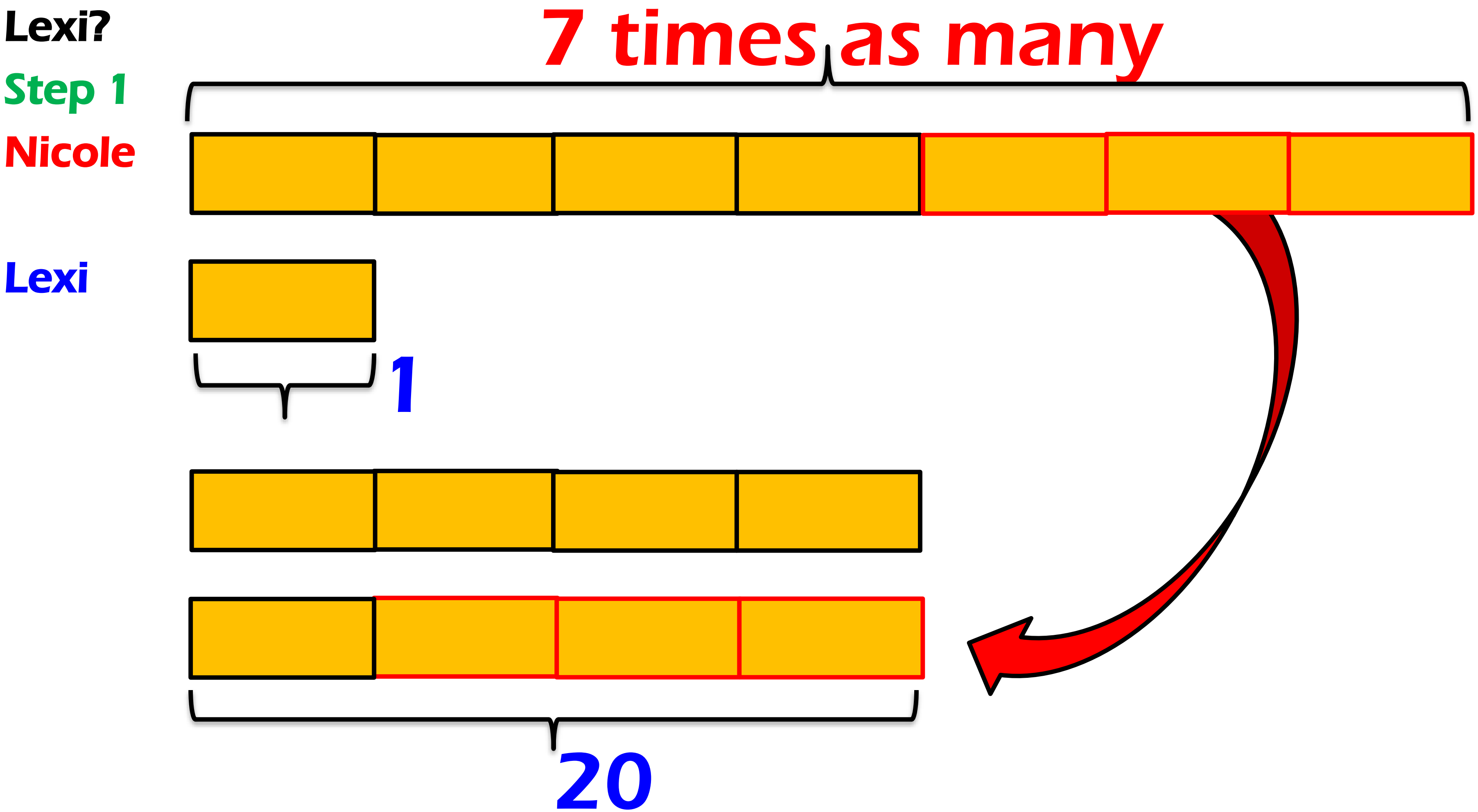

Another Way to Think About It or Represent It

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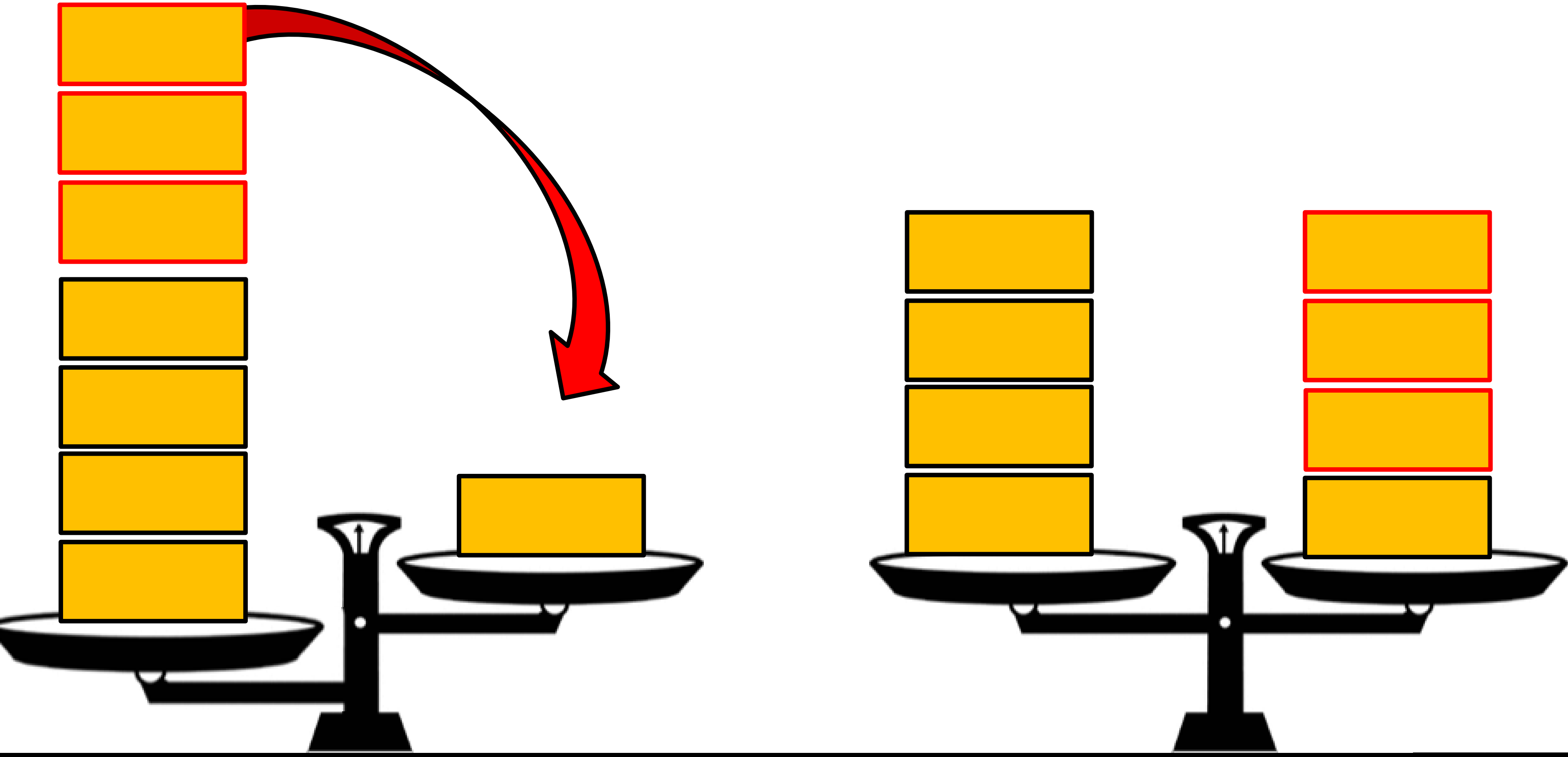
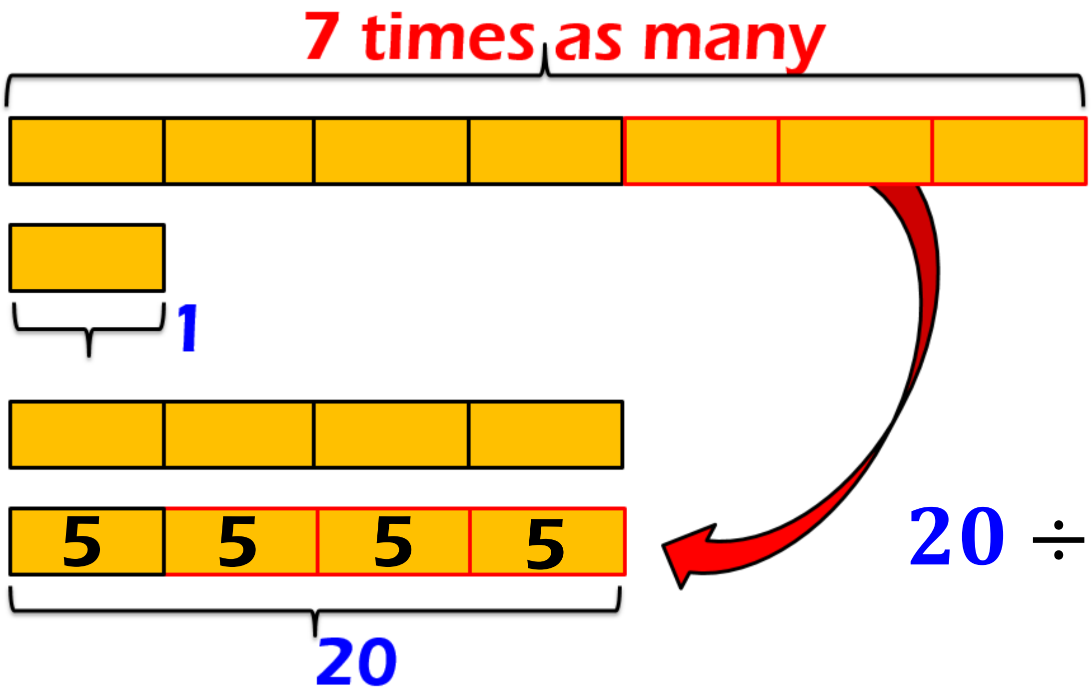


Transition to the Abstract Stage



Nicole had 7 times as many nickels as Lexi. After Nicole gave Lexi some of her nickels, each girl had 20. How many nickels had Nicole given to Lexi?

$$20 \div 4 = 5 \text{ or } 4(x) = 20$$

$$5(3) = 15$$

1. **Make them the same length, balance the number of boxes.**
2. **That length is 20.**
3. **Divide by the number of boxes to find the unit rate, nickels per box.**
4. **Use the unit rate to find the number of nickels given away.**

Abstract Stage

Nicole had 7 times as many nickels as Lexi. After Nicole gave Lexi some of her nickels, each girl had 20. How many nickels had Nicole given to Lexi?

There are 8 sets of nickels total.

$$7\ sets + 1\ set = 8\ sets$$

If they had the same amount, then they would each have 4 sets.

$$8\ sets \div 2\ people = 4\ sets\ per\ person$$

Twenty total nickels divided between 4 sets is 5 nickels per set.

$20 \div 4 = 5$ **or** $4(x) = 20$

Lexi started with 1 set and finished with 4 sets so the difference, what she was given, is 3 sets.

$$4 - 1 = 3$$

Lexi was given 3 sets with 5 nickels per set.

Lexi was given 15 nickels.

$$5(3) = 15$$

What is the value of the language included here?

Nicole had 7 times as many nickels as Lexi. After Nicole gave Lexi some of her nickels, each girl had 20. How many nickels had Nicole given to Lexi?

Available Equations:

$$N = 7L$$

$$L + g = 20$$

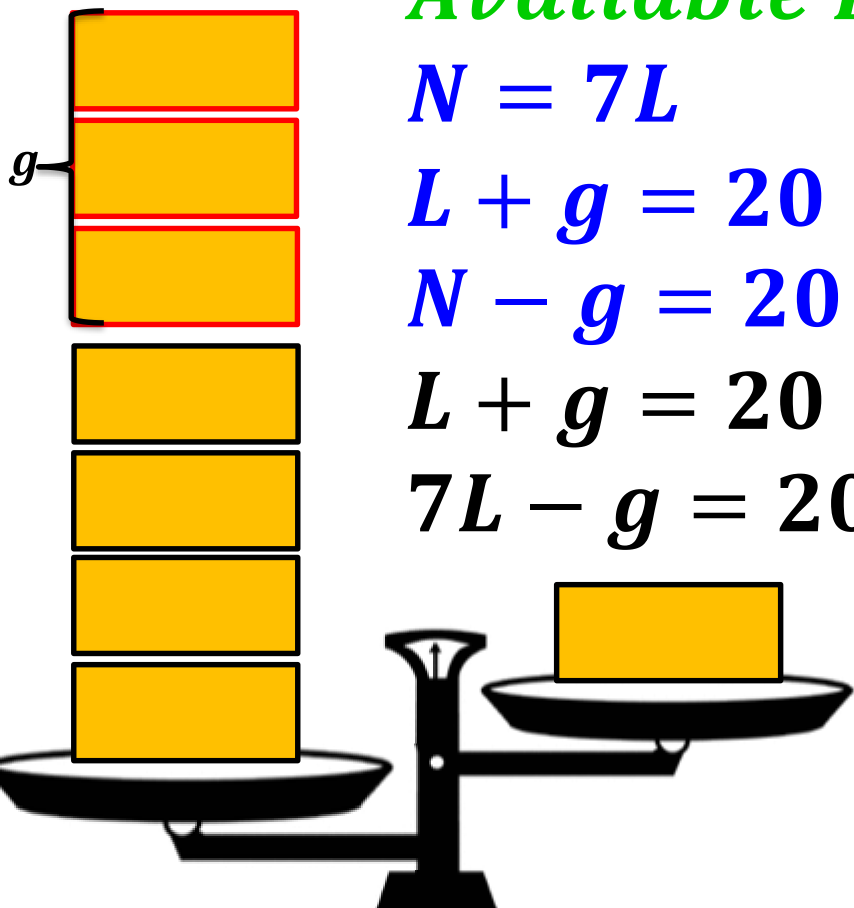
$$N - g = 20$$

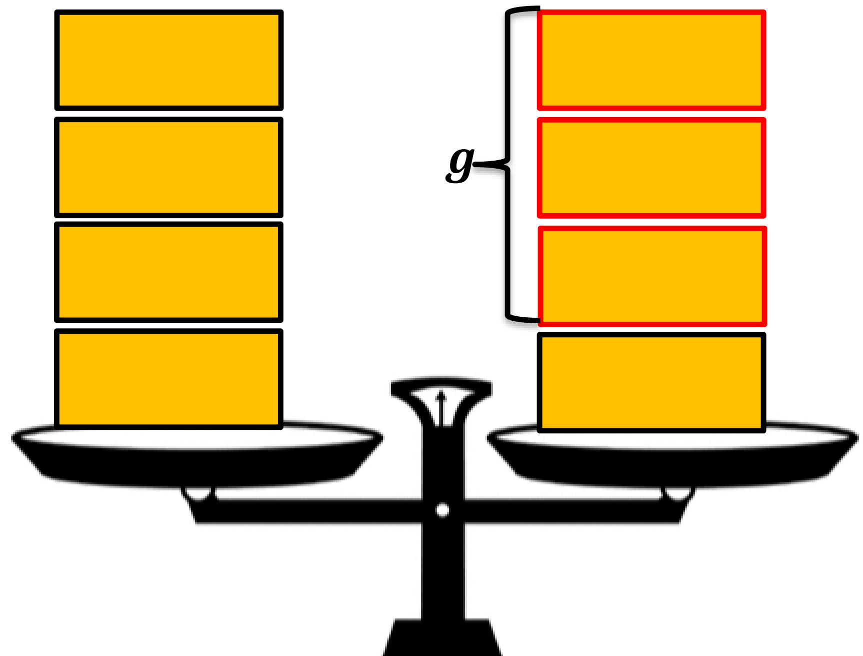
$$L + g = 20$$

$$7L - g = 20$$

$$(N - g) + (L + g) = 40$$

$$(7L - g) + (L + g) = 40$$





Solving by Substitution with Elimination

$$N = 7L$$
$$(N - g) + (L + g) = 40$$

$$(7L - g) + (L + g) = 40$$

Solving by Elimination with Substitution

$$N = 7L$$
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$$L + g = 20$$

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Handouts

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Multiple Representation Guide

Revised April 22, 2019

WORDS	PICTURES	SYMBOLS
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CONNECTIONS

DIRECTIONS: Write out the math scenario using words in the first column. Then, use the next two columns to (1) draw a picture of the math scenario, and (2) represent the scenario with symbols. **Finally, use arrows, circles, highlighters, text, and other annotations to show connections between the three boxes.**

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DEREK ROUCH
drouch@hemetusd.org

Mattoon's MEANING 4 MEMORY

Multiple Representation Guide

Revised April 22, 2019

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Levels of Abstraction = Concreteness Fading Concrete, Representational, Abstract

During the first half of my teaching career, I would spend what seemed to be the first half of a math lesson teaching a new math concept by **sharing definitions, formulas, steps and procedures.**

To make things more challenging for my students, **I would simultaneously introduce the symbolic notation** used to represent those ideas. Then, I would spend the remainder of the lesson **attempting** to help my students make sense of these very **new** and often **abstract ideas**.

By the end of the lesson, I could help many students build an understanding, but **there was always a group I felt who I would leave behind.** Like many other teachers, I was just **teaching in a very similar way to that how I was taught**. **I knew no different.**

However, if we consider that **new learning requires the linking of new information with information they already know and understand**, we should be intentionally planning our lessons with this in mind. A great place to start new learning is through the use of a **meaningful context** and **utilizing concrete manipulatives** that students can touch and feel.

When we teach in this way, we **minimize the level of abstraction** so students can **focus their working memory on the new idea** being introduced in a meaningful way.

- Kyle Pierce, Tap into Teen Minds, https://tapintoteenminds.com/concreteness-fading/

Connecting CRA